Null Sequences in the Theory of Limits

5apr11

1. Introduction

To help you to read Chapter 14 of our textbook, we take a novel approach. By introducing extra concept, **null sequences** , or just **nulls** , the subject becomes enormously simpler, especially the proofs and the solving the problems. It consists of an adaptation to an elementary level of the *Landau Symbols* of mathematical analysis. The concept carries over to continuous and differentiable functions, and even to a matrix based advanced calculus course.

1.1. Relaxed Notation

Note, by the way, that the text-authors have moved on to a more relaxed expository style, replacing "$\lim_{n\to\infty} a_n = L$" by "$a_n \to L$" . But they still revert to a primitive ε, δ argument in each proof. We will go a small step further. We shall agree that *null sequences* , i.e. sequences $\alpha_n \to 0$, are defined as usal, using ε, δ :

Definition of a Null Sequence

$$\alpha_n \to 0 \text{ means the same as } (\forall\ \varepsilon > 0)(\exists\ N)(\forall\ n > N)(\ |\ \alpha_n\ |\ \ < \varepsilon).$$

Definition of a Limit

$$a_n \to L \text{ means the same as } (\exists\ \alpha_n \to 0)(a_n = L + \alpha_n)$$

Henceforth, we shall use Roman letters for arbitrary sequences, but Greek letters means "null" even if we don't always say so.

1.2. Application to an Example

In the [Text, Lemma 14.3] has the hypotheses $a_n = L + \alpha_n$ and $b_n - a_n = \gamma_n$. So, adding, $b_n = L + (\alpha_n + \gamma_n)$. Done!

Of course, we need a theorem that says that "the sum of nulls is null" and so forth.

2. The Arithmetic of Null Sequences

For this concept to work for us, we need to know its rules of manipultion. We develop this here, but you can remember it simply as this: If a proposition makes sense and is true for the number 0 then it holds for null sequences. Just as $0 \pm 0 = 0$ and $0 \times k = 0$ for any constant, the sum of two or (finitely) many more null functions is null. So is their difference, and their products. Also, the product of a null sequence with a convergent sequence is null. Quotients are another matter, just as $0/0$ is forbidden.

Proof of [Text 14.5] Suppose $*$ stands for the operations $+$, $-$, $\times$ and maybe $\div$, but see below. Then we can compute

$$a_n * b_n = (L + \alpha_n) * (M + \beta_n) = (L * M) + \gamma_n$$

Where γ_n is null, being the sum, difference, and product of null functions.

Question 1.

Calculate γ_n from the equations, and identify each rule used to show that $\gamma_n \to 0$.

For the product, you'll need the distributive law. Then you'll need that a constant multiple of a null sequence is null, that the product of two null sequences is null, and the sum of any finite number of null sequences is null. These in turn follow nicely from the definition, using epsilons.

2.1. Quotients of Sequences

Let's see what you have to do with a quotient. The first step is always mechanical:

$$\frac{a_n}{b_n} = \frac{L + \alpha_n}{M + \beta_n} = \frac{L}{M + \beta_n} + \frac{\alpha_n}{M + \beta_n}.$$

The RHS has two summands, and they are treated differently. But first let's take care of the denominator. We need to assume that $M \neq 0$. So first we must make $n > N_1$ so that $(\forall\ n > N_1)(\beta_n < |M| / 2)$ where N_1 is guaranteed to exist by half the hypotheses (which half?). This keeps the denominators on the RHS from being zero by keeping it larger than $|M| / 2$.

We can now dispose of the *second* summand, by choosing an N_2 so that

$$(\forall\ n > N_2)(|\alpha_n| < \frac{|M|}{2}\frac{\varepsilon}{2}).$$

(What good is that?) This leaves the first summand and we're still not done.

We want to show that $\frac{L}{M + \beta_n} = \frac{L}{M} + \gamma_n$ for some null sequence γ_n . So solve for

$$\gamma_n = -\frac{L}{M}\frac{\beta_n}{M + \beta_n}.$$

Check my arithmetic. And now, pick a (possibly bigger) N_3 for β_n so that that $|\gamma_n| < \frac{\varepsilon}{2}$. (More arithmetic.)

Moral of the Story: The calculus of null functions is a nearly mechanical way of proving tricky limit statements.

Proof of [14.7]: This one is actually a statement about null functions and generalizes the idea of multiplying $0k = 0$ to a null function times a bounded function.

Question 2.
Show that if $\alpha_n \to 0$ and $|b_n| < B$ then $\alpha_n b_n$ is null. Hint: Choose the N_α for ε / B and complete the arithmetic.

Theorem [14.10]: If $0 \leq x < 1$ and $\frac{b_{n+1}}{b_n} = x + \beta_n$ then b_n is a null sequence.

Proof: This is an interesting application of two concepts. First, suppose that $b_n = L + \beta_n$, i.e. that the given sequence converges, and $L \neq 0$. Let's, for simplicity, assume everything is positive, which eliminates a lot of absolute value signs. Then substituting, and applying the rules of null functions, yields

$$\frac{b_{n+1}}{b_n} = \frac{L + \beta_{n+1}}{L + \beta_n} = \frac{L}{L + \beta_n} + \frac{\beta_{n+1}}{L + \beta_n} \to 1.$$

Note the denonminators are all bounded below by $L / 2$ once the $|\beta_n < L / 2$. That $L / 2$ was our epsilon for the "eventually". Since this contradicts the hypothesis, we're done. $L = 0$ and b_n is null. But we are not told to assume that b_n converges. That comes first. For $x < 1$ we can insure that $\frac{b_{n+1}}{b_n} < 1$ eventually. But multiplying through establishes the b_n to be a monotonically decreasing sequence bounded below by 0 .

Question 3.
Explain why the limit x of b_{n+1} / b_n needs to be in $[0, 1)$ for b_n to converge using the above argument.

3. Preview of Null Functions

[This section is optional and will not be tested on the final in MA348SP11.]

This brings us up to the high point of this chapter, the *Bolzano-Weierstrass Theorem* , which we'll take up in the next edition of these notes.

But anticipating Ch15, we can define *null functions* to be functions with this property:

$$(\forall\ \varepsilon > 0)(\exists\ \delta > 0)(\forall\ -\delta < h < 0 \text{ and } 0 < h < \delta)(\ |\ \alpha(h)\ |\ < \varepsilon)$$

The region $\{h\ |\ -\delta < h < 0 \text{ and } 0 < h < \delta\}$ is called a *punctured neighborhood* of 0 .

This allows us to work with the practical expression

$$f(a+h) = f(a) + \alpha(h) \text{ for } \lim_{x \to a} f(x) = L.$$

Exercise 1. See how many of the above theorems above about convergent sequences carries over, statement *and* proof, to functions.

A function is said to be *continuous* at x if in a punctured neighborhood of 0 for h we have that $f(x+h) = f(x) + \theta(h)$, for some null function $\theta(h)$.

A function is said to be *differentiable* at x , with derivative $f'(x) = m$, if in a punctured neighborhood of 0 for h we have that

$$f(x+h) = f(x) + (m + \theta(h))h$$

for some null function.

Exercise 2. See how many properties of derivatives you can now { alculate} directly from this definition.

If you'll do the sum and product rules of differentiation, then I'll do the chain rule, which is the most remarkable improvement over the usual exposition in standard calculus books.