PyBounce: A Physical Model of Bouncing Balls

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1 Introduction

This paper is a description of PyBounce, a project that has been completed as part of a summer research project for undergraduates in math and computer science. PyBounce is a physical model of bouncing balls, coded in the Python programming language[1].

This document explains the physical aspects of the model, along with issues that are likely to arise in any implementation. The first section explains the physical issues only, so that no computer programming experience is required. Familiarity with basic vector operations is assumed. A basic familiarity with high school level physics might also be helpful, although no specific knowledge of physics is assumed. Although Python code fragments are included in later sections for explanatory purposes, little knowledge of Python is actually required. The reader who has taken an introductory computer programming course should be adequately prepared for these sections.

2 The Physics of PyBounce

Finding the velocities of two balls after a fully elastic collision, given initial positions, initial velocities, and masses, is the central physical problem. The solution is written out below as a reference for anyone who would like to use it.

2.1 Background

A fully elastic collision between two balls here refers to a collision that preserves momentum and kinetic energy.

Stated mathematically, suppose we have two spherical balls B_1 and B_2 of masses m_1 and m_2 and of velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. The velocities after the collision are $\mathbf{v}_1'$ and $\mathbf{v}_2'$. With this notation, the fully elastic collision described above may be described mathematically by the following equations.

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[1]A Python tutorial can be found at www.python.org.

$$
\begin{aligned}
m_1\mathbf{v}_1 + m_2\mathbf{v}_2 &= m_1\mathbf{v}_1' + m_2\mathbf{v}_2' &(1)\\
\frac{1}{2}m_1\mathbf{v}_1\cdot\mathbf{v}_1 + \frac{1}{2}m_2\mathbf{v}_2\cdot\mathbf{v}_2 &= \frac{1}{2}m_1\mathbf{v}_1'\cdot\mathbf{v}_1' + \frac{1}{2}m_2\mathbf{v}_2'\cdot\mathbf{v}_2' &(2)
\end{aligned}
$$

The *radial components* of B_1 and B_2 are v_1 and v_2. This term is used here to refer to the component of velocity that is affected by the collision (see Figure 1).

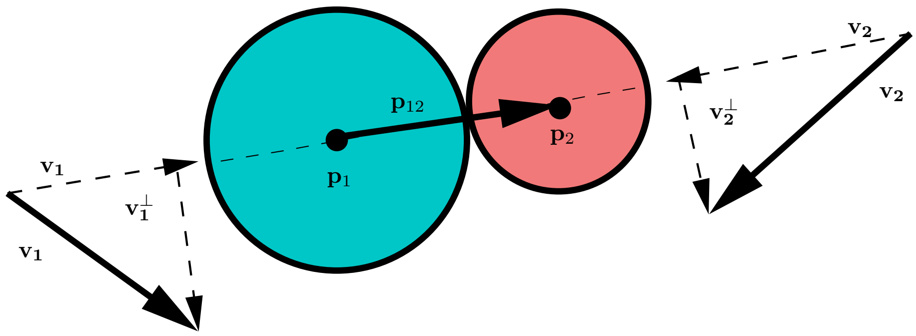

Figure 1: Velocity Components

More precisely, v_1 and v_2 are described as follows. To begin with, let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the positions of B_1 and B_2 (that is, the positions of their corresponding centers). Let $\mathbf{p}_{12} = \mathbf{p}_2 - \mathbf{p}_1$, so that $\mathbf{p}_{12}$ is the vector pointing from p_1 to p_2. To simplify matters, let $\mathbf{u} = \frac{\mathbf{p}_{12}}{|\mathbf{p}_{12}|}$. Now, given $\mathbf{v}_i$, where i=1 or i=2, we let $v_i = \mathbf{v}_i \cdot \mathbf{u}$. In other words, v_1 and v_2 are the scalar projections of the velocities of B_1 and B_2 onto the vector pointing from B_1 to B_2.

By resolving the velocities into components in this way, we are able to reduce the general situation depicted in figures 2 and 3 below to that of a head on collision. This is done as follows.

Let $\mathbf{v}_1^\perp = \mathbf{v}_1 - v_1\mathbf{u}$ and $\mathbf{v}_2^\perp = \mathbf{v}_2 - v_2\mathbf{u}$. These vector components of velocity are perpendicular to $\mathbf{u}$ and are unaffected by the collision. Their contribution to momentum and kinetic energy is therefore the same before and after the collision[2]. By using equations 3 through 6 below, we may therefore reduce equations 1 and 2 to equations involving only radial components. Put differently, the reduced equations will model a head on collision.

$$
\begin{aligned}
\mathbf{v}_1 &= v_1\mathbf{u} + \mathbf{v}_1^\perp &(3)\\
\mathbf{v}_2 &= v_2\mathbf{u} + \mathbf{v}_2^\perp &(4)\\
\mathbf{v}_1' &= v_1'\mathbf{u} + \mathbf{v}_1^\perp &(5)\\
\mathbf{v}_2' &= v_2'\mathbf{u} + \mathbf{v}_2^\perp &(6)
\end{aligned}
$$

[2]We ignore friction and angular momentum.

To obtain these reduced equations, we first substitute the expressions for $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}'_1$, and $\mathbf{v}'_2$ above into equation 1, which gives the first equation below. We then perform a little simplifying algebra.

$$m_1(v_1\mathbf{u} + \mathbf{v}_1^\perp) + m_2(v_2\mathbf{u} + \mathbf{v}_2^\perp) \quad = \quad m_1(v'_1\mathbf{u} + \mathbf{v}_1^\perp) + m_2(v'_2\mathbf{u} + \mathbf{v}_2^\perp)$$

After expanding both sides of the equation and subtracting from both sides each term including a perpendicular component, we obtain the equations below.

$$\begin{aligned} m_1 v_1\mathbf{u} + m_2 v_2\mathbf{u} &= m_1 v'_1\mathbf{u} + m_2 v'_2\mathbf{u} \\ (m_1 v_1 + m_2 v_2)\mathbf{u} &= (m_1 v'_1 + m_2 v'_2)\mathbf{u} \\ (m_1 v_1 + m_2 v_2)\mathbf{u} - (m_1 v'_1 + m_2 v'_2)\mathbf{u} &= 0 \\ ((m_1 v_1 + m_2 v_2) - (m_1 v'_1 + m_2 v'_2))\mathbf{u} &= 0 \end{aligned}$$

Since $\mathbf{u} \neq 0$, we obtain the desired equation.

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2$$

The reduced equation for conservation of kinetic energy can be obtained similarly, so that we have the following system of equations.

$$\begin{aligned} m_1 v_1 + m_2 v_2 &= m_1 v_1^{'} + m_2 v_2^{'} \qquad (7) \\ \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 &= \frac{1}{2} m_1 v_1^{'2} + \frac{1}{2} m_2 v_2^{'2} \qquad (8) \end{aligned}$$

We will solve these equations for v'_1 and v'_2 in the next section. Once we have these values, we will be able to compute $\mathbf{v}'_1$ and $\mathbf{v}'_2$ according to equations 5 and 6, which is the ultimate aim of this paper.

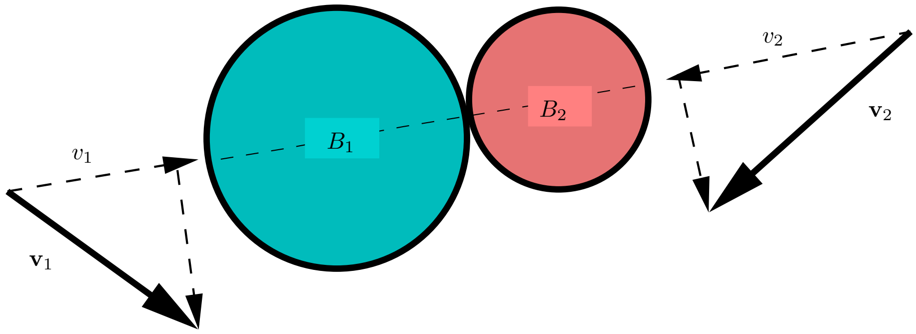

Figure 2: Before Collision

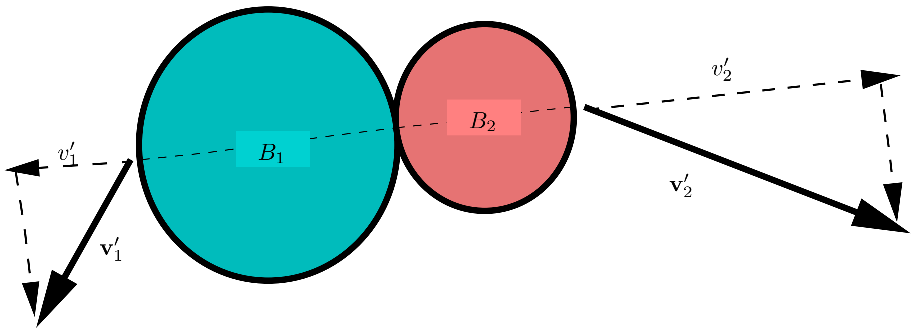

Figure 3: After Collision

2.2 Solving the Equations

The masses, positions, and initial velocities are known, so we know m_1 and m_2 immediately, and we can calculate v_1 and v_2 according to the last section. We thus have two equations in two unknowns. It would be nice to have two linear equations in two unknowns, so the first thing to do is rewrite equations 1 and 2 in the forms below.

$$\begin{aligned} m_1(v_1 + v_1')(v_1 - v_1') &= m_2(v_2 + v_2')(v_2' - v_2) \\ m_1(v_1 - v_1') &= m_2(v_2' - v_2) \end{aligned}$$

We know that $v_1 \neq v_1'$. Otherwise, we would have $v_1 = v_1'$ and $v_2 = v_2'$, as the masses are assumed to be nonzero, which would mean the two masses pass through each other. Therefore, by dividing, we obtain the following from the equations above.

$$v_1 + v_1' = v_2 + v_2'$$

Rewriting this equation in a slightly different form and combining with equation 7 gives the two linear equations below.

$$\begin{aligned} v_1' - v_2' &= -v_1 + v_2 \qquad (9) \\ m_1 v_1 + m_2 v_2 &= m_1 v_1' + m_2 v_2' \end{aligned}$$

All that is left is to solve these equations. Matrices are used here, but if the reader is unfamiliar with matrix operations, any other method will of course do just fine.

$$\begin{pmatrix} 1 & -1 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$
\begin{aligned}
\begin{pmatrix} 1 & -1 \\ m_1 & m_2 \end{pmatrix}^{-1} &= \frac{1}{m_1+m_2}\begin{pmatrix} m_2 & 1 \\ -m_1 & 1 \end{pmatrix} \\
\begin{pmatrix} v_1^{'} \\ v_2^{'} \end{pmatrix} &= \frac{1}{m_1+m_2}\begin{pmatrix} m_2 & 1 \\ -m_1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ m_1 & m_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} v_1^{'} \\ v_2^{'} \end{pmatrix} &= \frac{1}{m_1+m_2}\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} v_1^{'} \\ v_2^{'} \end{pmatrix} &= \frac{1}{m_1+m_2}\begin{pmatrix} (m_1-m_2)v_1 + 2m_2v_2 \\ 2m_1v_1 + (m_2-m_1)v_2 \end{pmatrix}
\end{aligned}
$$

After multiplying through by $\frac{1}{m_1+m_2}$, we obtain the solution below.

$$
\begin{aligned}
v_1^{'} &= \frac{m_1-m_2}{m_1+m_2}v_1 + \frac{2m_2}{m_1+m_2}v_2 && (10) \\
v_2^{'} &= \frac{2m_1}{m_1+m_2}v_1 + \frac{m_2-m_1}{m_1+m_2}v_2 && (11)
\end{aligned}
$$

Equivalently, if we let $\bar{m}$ and s_m be the average and the semi-difference of the masses,

$$
\begin{aligned}
\bar{m} &= \frac{m_1+m_2}{2} \\
s_m &= \frac{m_1-m_2}{2}
\end{aligned}
$$

then the solution below is equivalent.

$$
\begin{aligned}
v_1^{'} &= \frac{s_m}{\bar{m}}v_1 + \frac{m_2}{\bar{m}}v_2 \\
v_2^{'} &= \frac{m_1}{\bar{m}}v_1 + \frac{-s_m}{\bar{m}}v_2
\end{aligned}
$$

Lastly, we return to equations 5 and 6.

$$
\begin{aligned}
\mathbf{v}_1' &= (\frac{s_m}{\bar{m}}v_1 + \frac{m_2}{\bar{m}}v_2)\mathbf{u} + \mathbf{v}_1^{\perp} && (12) \\
\mathbf{v}_2' &= (\frac{m_1}{\bar{m}}v_1 + \frac{-s_m}{\bar{m}}v_2)\mathbf{u} + \mathbf{v}_2^{\perp} && (13)
\end{aligned}
$$

Given the masses, initial positions, and initial velocities of two spherical balls undergoing a fully elastic collision, we have now solved for the outgoing velocities.