Kepler's Laws, Universal or Not?

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Abstract

While it may seem as though the heavens are an unorderly place, each and every planetary body actually follows the same set of patterns. These patterns were at one point only theory, but they now exist as universal laws. However, this project will focus exclusively on the planets within our solar system.

The first law states that the orbit of a comet or planet is always conic in shape. Whether an ellipse, hyperbola, or parabola, each planetary motion can be modeled and predicted by the mathematics of each type of path. The second law exists as an extension of the first, but is dependent on the line segment connecting the planet to the sun. It states that this line segment sweeps over equal areas in equal periods of time. Finally, the third law only holds true for objects with closed, or elliptical, orbits. This final law relates the square of the orbital period to the cube of the semimajor axis of the orbit. Luckily for us, every planet within our solar system follows a closed orbit and we can therefore illustrate the third law with the same planetary bodies as will be used in the first two parts.

1 Introduction and Background

Throughout the early years of astronomy, it was believed and accepted that the earth was located at the center of the universe. Based partially in science and partially in religion, this claim went virtually unquestioned for thousands of years. However, in 1543 Nicolaus Copernicus published the extremely contraversial idea that this claim was actually false. He proposed the idea of heliocentricity, or the notion that the universe was centered around the sun, not around the earth. Between the years of 1609 and 1619, Johannes Kepler published a work which would strongly support this idea of heliocentricity. The work contained what would later be called Kepler's Laws of Planetary Motion, and went on to explain and describe the motions of the heavenly bodies. My goal with this project is to model these three laws, illustrating the motions of the planets while also mathematically proving them. All three laws will be dependent on the law of universal gravitation, which Newton defined as:

$$F = G\frac{Mm}{r^2}$$

In this case, M is the mass of the sun, m is the mass of the planet, r is the distance between these two objects, F is the force that they exert on each other, and G is the constant of gravitation. Using this equation to determine the effects that each planetary body has on one another, their respective motions can not only be calculated, but easily illustrated. While the math behind these models will be based only on proven calculations, not all of these proofs will be provided. Any steps that are skipped over will be covered in much more detail during the seminar.

2 Kepler's First Law

A planet revolves around the sun in an elliptical orbit with the sun at one focus.

2.1 Underlying Math

After finding the equation of the orbit, proving this first law is actually relatively simple. The derived equation of orbit written in polar coordintes is:

$$\frac{1}{r} = \frac{1 + e * cos(\theta)}{p(1+e)}$$

In this case, p represents the perihelion which is equivalent to the smallest value of r, or the shortest possible distance beteen the planet and the sun. We can then manipulate the function in the following way:

$$p(1+e) = r(1+e*cos(\theta))$$

$$p(1+e) = r + ex$$

$$r = e[p(1+\frac{1}{e}) - x]$$

Where r is the the planets distance from the sun, and the expression in brackets represents the the perpendicular distance from the line D, whose x coordinate is $x = p(1+\frac{1}{e})$. This affirms that the planet lies on a conic shape with a focus at the origin(the sun), an eccentricity of e, and a directrix of D.

2.2 Program Details

This model will be relatively simple in the sense that only two objects need to be created at any given time. At least three examples will be illustrated, each using the same sun but with varying sizes of planets. The program will ask for user input, and this input will be used to decide which planets or combination of planets will be displayed. VPython allows for the easy creation of each of the celestial bodies. After bestowing the size, position, and physical features if each object, the universal law of gravitation will be used to determine the vector quantities of the forces that are exerted. Updating each object's momentum by applying this force at various points in time, the movement of the planets around the sun will be demonsrated with relative scale.

3 Kepler's Second Law

The line segment joining the sun to a planet sweeps out equal areas in equal times.

3.1 Underlying Math

In order to determine the area swept out by the line segment, we can actually treat the area as a triangle. The line segment itself makes up the larger leg, the change in position makes up the smaller leg, and the hypotenuse connects the planets new position to the sun. While this may seem inaccurate due to the fact that the smaller leg will be a curve and not a straight line, we can change this by making the movement of the planet infitesmially small. By doing this, the shape of the area approaches that of a triangle, and this is what we will use for the calculations. The change in area can then be represented by:

$$dA = \frac{1}{2}r^2\Delta\theta dt$$

Where r is the distance from the sun to the planet, and $r\Delta\theta$ is the change in position of the planet. The areal velocity, or rate at which area is swept, we then define as:

$$K = \frac{1}{2}r^2\Delta\theta$$

Differentiating with respect to time:

$$K' = \frac{r}{2}(r\theta'' + 2r'\theta')$$

$$K' = 0$$

Previous calculations have allowed us to determine that the quantity in parenthesis has the value of 0, and therefore K is a constant. This means that area is swept out at a constant rate so equal areas are swept in equal times.

3.2 Program Details

A substantial amount of code from the previous model will be reused for the second law. This will still provide an accurate representation of the movements of the planets around the sun, and focus can then be shifted to showing the areas that are acccumulated. I plan to take advantage of the make trail function in order to show the amount of area covered over acertain period of time. This will be done by creating a line segment to connect the two objects, and then creating the trail on this line. Another addition to the code will be a display or some sort of print feature that will show the current area that has been covered. For example, a bar graph may be created with the y axis representing area and each mark on the x axis representing a new, but equal, interval of time. As time ellapses for each given interval, the bar will increase until that interval is over. After the planet has made one rotation, every one of the bars will have an equal height, and therefore equal areas will have been covered. Such a feature is being added in order to counter the fact that estimating the area of a non-regular shape can be hard to do by merely looking at it.

4 Kepler's Third Law

The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

4.1 Underlying Math

Using the equation of the orbit that was derived for the first law, we can determine that the semiaxis in the x direction, denoted by a, is equal to the value $\frac{p}{1-e}$, which can then be rewritten as $p = a(1 - e)$. Through substitution and manipulation*, we arrive at the equation of the orbit in rectangular coordinates:

$$(\frac{x + ae}{a})^2 + (\frac{y}{a\sqrt{1 - e^2}})^2 = 1$$

Newton's law of universal gravitation can be related to the perihelion (minimum value of r) in the following way:

$$p = \frac{4K^2}{GM(1 + e)}$$

This is later manipulated* until we arrive at a relationship that supports, and actually is, Kepler's third law:

$$T^2 = \frac{4\pi^2}{GM}a^3$$

* As stated in the intorduction, the actual steps taken in this and the previous two parts will be explained in much more detail during the seminar next week.

4.2 Program Details

Once again, a majority of the code from the first law will be used as the foundation for this law. This way, the same visual representations will be in effect, however, a display or print function will be added in order to display both the length of the major axis as well as the period of the orbit. The program wiould then allow the user to input the desired length of the major axis. Using this input in its calculations, the period of revolution will be displayed alongside the users original input and the visual representation will be adapted in order to accurately model the result.

5 Projected Timeline

24 Oct 2016 : The proposal and seminar will be finished; Focus can then be shifted towards writing the code and documentation

31 Oct 2016 : Finished with the rough draft of the first law; The first draft of the second law (without graph) will be completed as well

7 Nov 2016 : A second draft of the second law (with graph) and first draft of the third law will be finished

14 Nov 2016 : Finished with a working rough draft of all three laws

21 Nov 2016 : All asthetic touches will be finished

28 Nov 2016 : The website will be fully updated and customized