Explanation of The Westergame

Alec Mori

December 2013

1 Introduction

While bracing a quasicrystal may seem like an abstract geometric concept to most, a 2D quasicrystal can be solved using simple graph theory and a bit of abstract thinking. Firstly, a quasicrystal is "a planar arrangement of rhombi, which is [...] simply connected."[1] This figure should contain no holes and have no two rhombi overlap; between any two rhombi there exists a path of rhombi. For easier computation, any angle θ that an edge holds from the horizontal will be defined such that

$$\{\theta \mid 0 \leq \theta < 2\pi, \theta \in \{0, 2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5\}\} \tag{1}$$

Although it is not inherent in the quasicrystal framework, in order to more easily manage the graphing and solving of the graph, we must introduce a concept known as ribbons. A ribbon runs through a face, intersecting the midpoints between two parallel edges of a given rhombus and running parallel to the other two sides. Each ribbon begins at one end of the framework and terminates at the other end.

A rhombic plane (including this quasicrystal) can be altered by shifting the angles of all rhombi such that the plane is still simply connected. A rhombic plane has a "rigid bracing" if enough of the faces are "braced" (cannot have its angles shifted), so that no faces, even if they are unbraced, can have their angles shifted.

2 Solving the Wester's Theorem

Let's define some graph γ so that all vertices (V) in γ are ribbons in the quasicrystal and all edges (E) in γ are the intersections between said ribbons. We can also define ψ as a braced subgraph of γ. We now pick arbitrary ribbons $a, b, c \in V$ and define three angles α, β, δ such that

$$\alpha = \angle ab, \quad \beta = \angle bc, \quad \delta = \angle ac.$$

where α, β, δ are all acute angles. Now, because we are in a Euclidean plane, we know that the sides of a triangle must add up to π, so if two angles remain unmoved after a distortion of space around them, we know that the third must also remain unchanged afterwards.

[1] Duarte, 1.

The only possible transformation that may have occured is a rotation, which maintains the bracing (Proof of the Transitivity Lemma).

Now, let's assume we have some face ab (named after ribbons a, b) that is in some simply connected subgraph ξ. To check if the subgraph braced, we must ensure all edges in it are braced. Assuming that face ab is not in the graph (thus causing edge ab to not be in the subgraph), we know there exists a series of ribbons n that connect a and b, where

$$n \in a, r_1, r_2, r_3, \ldots, r_n, b.$$

Because the subgraph ξ is simply connected, we know that the ribbons are each connected to the adjacent ones, such that all faces

$$(ar_1), (r_1 r_2), (r_2 r_3), \ldots, (r_n b)$$

are braced. Which implies all angles

$$\angle ar_1, \angle r_1 r_2, \angle r_2 r_3, \ldots, \angle r_n b$$

are fixed. And because of the transitivity lemma, we see $\angle ar_2$ holds, and with a few more recursions, we see that $\angle ab$ will be braced, which implies the subgraph is braced.

3 How to Represent This Theorem

The best way to display and analyze this theorem is to display each ribbon r in the set of all ribbons R as a node in graph ε and each braced face $r_1 r_2$ as an edge between ribbons r_1 and r_2. If the graph is connected (there exists a path between all nodes in the graph), then the quasicrystal is considered braced - if the graph is connected with the fewest number of edges, then the quasicrystal is considered minimally braced.

In order to check if a given quasicrystal state is braced, we must perform a breadth first search. First we take an arbitrary node (in practice, we always start with the one with ribbon zero) and mark it as "Visited". Then, we follow all edges along said node, and add any node we find not marked "Visited" to a list, which we then perform the same search on. We do this until we run out of nodes not marked "Visited" connected to our initial point.

In order to be able to brace faces to create the edges, we must also have the quasicrystal displayed on the screen as well. Because it is a face of rhombuses, each face can be portrayed as four points and four edges, all of equal magnitude. If we were to turn each edge into a linear line $y = mx + b$, it is fairly trivial to get the mouse point and then check if it lies between all four lines that make up the rhombi's edges, and easier still to select/deselect the face accordingly. Because each ribbon could also be represented by the line $y = mx + b$, it also is easy to find out which ribbon we are closest to, by finding the distance from that line to the clicked point and then making sure the ribbon is localized near that point.

Bending, or shearing, of the quasicrystal along a ribbon is done in a very similar manner to looking through the graph of all ribbons r in R – a breadth first search. If a face is braced, we brace those around it as stated in the Wester Theorem, and add all points to the queue necessary to perform the next layer of a breadth first search.

References

1. Eliana M. Duarte Gélvez, George K. Francis. *Stability of Quasicrystal Frameworks in 2D and 3D.* 2013. http://new.math.uiuc.edu/quasistable/DuarteFrancisSeville1may13.pdf

2. Chris Caldwell. *Graph Theory Glossary.* 1995. http://www.utm.edu/departments/math/graph/glossary.html

3. *Breadth-first search.* 2013. http://www.princeton.edu/~achaney/tmve/wiki100k/docs/Breadth-first_search.html