1. Chapter introductions

1.1. Introduction

This work is about embedding Riemann surfaces conformally into $\mathbb{R}^3$. A *Riemann surface* is a one dimensional complex manifold, that is, a topological manifold, S, with an equivalence class of holomorphic atlases. Two atlases for S define the same Riemann surface, W, if and only if the composition of charts across the two atlases remain holomorphic. The only boundary allowed are finite sets of points, or *punctures.* Consider a Riemann surface, W, of genus p and with n punctures. We say that (p, n) is the *signature* of W. Thus, the signature is a way to denote the topological type of W without regard to its analytic structure.

A *dissection* of W is a series of cuts made in W along arcs, based at a single point, $v \in W$, such that the resulting surface with boundary, W', is simply connected. In particular, if the arcs all lie on geodesics, and we can always arrange that they do, W' is conformally equivalent to a polygon, $\mathscr{P}$, with congruent pairs of edges. Each cut in W is made along an open arc from v to one of the n punctures or is made along a closed loop based at v. The loops can be pictured as arcs winding around the various torus "handles" of W. A *marking* on W is given by an equivalence class of dissection cuts. Let D_v denote a set of cuts on W based at the point v. Two sets of cuts, D_v and D_u, are equivalent, as dissections, when each cut along a closed loop in D_v, is homotopy equivalent to one and only one such cut in D_u. A marked Riemann surface determines a point in $\mathrm{T}(p, n)$, the Teichmüller space for all marked Riemann surfaces with signature (p, n). In general, each Riemann surface, W, with signature (p, n), is represented by an infinite number of points in $\mathrm{T}(p, n)$, each point corresponding to a distinct marking of W.

Each Riemann surface is covered by one of the three universal covering spaces, namely, the Riemann sphere (S^2), the plane ($\mathbb{C}$), and the disc (D). Except for these universal covering spaces and the punctured plane, all other Riemann surfaces are conformally equivalent to a polygon lying in either D or $\mathbb{C}$. In 1883 Poincaré described in [P1] all polygons, $\mathscr{P}$, that are conformally equivalent to Riemann surfaces. One requirement is that the polygon $\mathscr{P}$ have edges identified in pairs. Suppose that a and b are a pair of identified edges in $\mathscr{P}$. Then there exists an isometry, f, of the universal covering space such that $f(a) = b$ with the correct orientation. We call such an f a *side-pairing* or *deck transformation.* The set of side-pairing transformations for $\mathscr{P}$ generate a group, G, which is isomorphic to the fundamental group of W, i.e., $G \cong \pi_1(W)$. The polygon $\mathscr{P}$ tiles the universal covering space of W under the action of G. Roughly speaking, a pair of identified edges of $\mathscr{P}$ correspond to an arc in a dissection on W. Such a polygon $\mathscr{P}$ is, in a sense, the result of cutting the surface W so that it will lie "flat" in its universal covering space.

In 1966 Keen developed a construction of canonical polygons, each of which represents a distinct marking of a Riemann surface with signature (p, n) [Ke1]. Arc and angle measurements in the polygons of Keen provide an intrinsic parametrization of the Teichmüller space, $\mathrm{T}(p, n)$. There are other schemes for parametrizing $\mathrm{T}(p, n)$. For example Abikoff [Ab] uses the trace values of certain elements of G, the deck transformation group of W. These trace values are called *Fricke parameters* for the Teichmüller space, $\mathrm{T}(p, n)$. The works of Keen and of Abikoff provided much of the motivation for video tapes described in Chapter 6 of this work.

In 1961 and 1971, respectively, Garsia [G2] and Ruëdy [R1] established that every closed

Riemann surface, respectively, arbitrary Riemann surface can be conformally embedded in $\mathbb{R}^3$. The works were not constructive and only isolated examples of conformal embeddings existed, such as the Clifford tori. In 1985 Pinkall [Pi] produced examples of conformal embeddings of all genus one Riemann surfaces, called *Hopf tori*, since the embeddings are based on the Hopf fibration of S^3.

1.2. Chapter 2

Hyperbolic isometries, the uniformization theorem, and the theory of Riemann surfaces

We begin with a discussion of hyperbolic isometries, presenting two equivalent ways of defining the action of $PSL(2,\mathbb{C})$ on the upper half-space, each making $PSL(2,\mathbb{C})$ isomorphic to the isometry group of hyperbolic 3-space, $\mathbb{H}^3$. We describe how the action of the *modular group* on the parameter space of closed, genus one Riemann surfaces, T(1,0), preserves conformal structures. Closed genus one is, essentially, the only non-trivial, non-hyperbolic case of Riemann surfaces and yet the hyperbolic plane has a presence even in this case. We present a brief synopsis of the uniformization theorem, focusing on the hyperbolic case. Finally, we present definitions of concepts used in the remainder of this work.

1.3. Chapter 3

Garsia, Rüedy, and Pinkall: Conformal embeddings of genus 1 Riemann surfaces in $\mathbb{R}^3$

This chapter grew directly out of a project suggested by A. Marden. We know that all Riemann surfaces can be conformally embedded into $\mathbb{R}^3$. The works of Garsia [Ga1] and of

Rüedy [Ru1] give us this result. However, there are few explicitly constructed examples of such embeddings. Pinkall constructed examples of genus-one embeddings. Unfortunately his construction is based on the Hopf fibration of S^3, which does not generalize to the higher dimensions required for the parameter (Teichmüller) spaces of higher genus Riemann surfaces[1]. We will show how to construct conformal embeddings for all *flat tori* that are conformally equivalent to rectangles with opposite edges identified.

1.4. Chapter 4

The Bishop and Frenet frames: shaping a space curve by deforming its development

An RTICA was designed to aid in shaping space curves by varying the parameters of their two canonical framings. We briefly describe these framings: the Frenet frame, which is classic, and the Bishop frame, which was introduced as an alternative in 1975 by R. Bishop [Bi]. We discuss parametrized tubular surfaces using *relatively parallel fields* defined by the frames of simple, closed space curves. We plan to construct these curves using the RTICA described in this chapter. This could generalize the result of Chapter 3 and lead to computed constructions of conformal embeddings for all *flat tori.*

1.5. Chapter 5

Möbius transformations in $\mathbb{H}^2$ and Jørgensen's classification of loxodromic transformations in $\mathbb{H}^3$

The loxodromic transformations characterize the difference between hyperbolic 2- and

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[1] The Hopf mapping allows us to have closed loops associated with each point in S^2. Thus, a curve, γ, which is homotopic to the equator on S^2 will naturally generate a torus in S^3. Thus, two pairs of edges are neatly glued together. With higher genus Riemann surfaces, there are $2(p-2)$ additional edges to identify.

3-space, $\mathbb{H}^2$ and $\mathbb{H}^3$. They act as *spiral dilatations,* which provide the most interesting motions within $\mathbb{H}^3$. However, loxodromic transformations are the isometries of $\mathbb{H}^3$ that can never be *Poincaré extensions* of isometries of $\mathbb{H}^2$; they fail to preserve any disc.

Möbius transformations acting on $\mathbb{C} \cup \{\infty\}$ can be classified in more than one way. We include classifications based on trace values, *Steiner nets,* elementary Euclidean transformations, and fixed points together with *isometric circles.*

Jørgensen gives us a particularly interesting partition of the trace parameter space of $PSL(2,\mathbb{C})$, which captures details of the fundamental polyhedra of cyclic Möbius transformation groups. The faces of these hyperbolic polyhedra lie on the *isometric spheres* of transformations that are cyclic group elements. Triple intersection points of isometric spheres provide a striking feature and determine the boundaries of the partition of the parameter space described by Jørgensen.

1.6. Chapter 6

Real-time interactive computer animators (RTICA)

Running roughly parallel with our studies of Riemann surfaces was the construction of the RTICA. Each RTICA was written as an instrument to illustrate known results, to visually investigate a particular question, or both. Writing the code for the RTICA and viewing the resulting graphics, led to the conformal embeddings of rectangular flat tori, the results in Chapter 3. The RTICA presented in Chapter 4 is an instrument designed to extend these results.

This Chapter contains descriptions of all RTICA constructed by the author of this work. The RTICA are based in the graphics tool, the *illiShell,* developed by G. Francis and several of his students at UIUC. The *illiShell* allows the RTICA to provide interactive animations on all GL and most OpenGL (platforms). With minimal adjustments, the RTICA can all be presented in each of the virtual reality environments (hardware settings) at the National Center for Supercomputer Applications (NCSA.) These include the hardware for CAVE^{TM}, ImmersaDesk^{TM}, and InfinityWall^{TM}, which are all trademarked by the Board of Trustees of the University of Illinois. This equipment is available at the NCSA facilities at the Beckman Institute on the UIUC campus. Compatible virtual reality environments are available at sites across the nation. Description of some currently available environments is given in [HMF]. CAVE-ready RTICA could be presented in these venues as well as in those at the NCSA facilities.

1.7. Chapter 7

Diagrams, videos, and virtual reality

The diagrams and illustrations herein were constructed using either *pic* or *Mathematica* as programming language. In some cases, existing *Mathematica* subroutines were rewritten in the *pic* language and processed through *gpic* software. Included as an example is the group of subroutines written in *pic* to compute the action of the Poincaré extension of Möbius transformations on points in hyperbolic 3-space. This code was used for illustrations in Chapter 3.

In each of the past two years, a ten minute video tape was produced for presentation at the Winter Joint Mathematics Meetings of the AMS, MAA, ALS, AWM, and NAM. In this chapter we present the narration that accompanied each video.

The CAVE is a visual and sound environment where three dimensional images are produced through computer imaging. Three walls and the floor are screens for four high-resolution graphics projectors. When an image "moves" from one screen (wall) to another, the powerful multiprocessor SGI Onyx machine coordinates the screen images so that the object appears to move through space with no seams or screen edges. The 3-dimensionality is accomplished by a process of double imaging. Infrared synchronized liquid crystal shutter glasses blink, so as to trick the viewer's eyes into converging at the required/expected angle of the object's virtual location. One truly feels that one should be able to feel the object when a hand has been reached out to where the object appears to be. One also feels the necessity to brace one's knees for a jolt as one "falls" through Euclidean space or is spun about in hyperbolic space in the CAVE RTICA.