Real-time Interactive Gravitational Lensing

George K. Francis
John S. Estabrook
Ulises Cervantes-Pimentel
Birgit Bluemer
Department of Mathematics
University of Illinois
gfrancis@uiuc.edu
http://www.math.uiuc.edu/∼gfrancis/

14 September 2000

Abstract

Our project simulates, in a real-time interactive environment, the general relativistic effect of the deflection of light by a massive object, in accordance with the standard Einstein model. This provides a qualitatively accurate representation, by way of a quantitatively reasonable approximation, of the phenomenological properties of a massive object acting as a gravitational lens, including distortion of the perceived environment, multiple imaging, and the Einstein ring.

Our project provides a versatile platform for investigation of lensing phenomena generally, and a timely research tool for the specifics of gravitational lensing. Moreover, the unique characteristic of viewpoint-based calculation, integral to this kind of simulation, offers a new class of benchmarks and diagnostics for multi-viewer virtual environments served by the *DuoView* extensions to the CAVE libraries, and the VPS cave-to-cave networking protocols.

Objectives

The objective of this project was to design and execute a series of mathematical experiments on a variety of immersive virtual reality platforms at the University of Illinois Electronic Visualization Laboratory (EVL), the National Center for Supercomputing Applications (NCSA), and the Imaging Technology Group (ITG) of the Beckman Institute. The purpose of the experiments was to explore the cosmological phenomenon known as "gravitational lensing" with real-time, interactive (CAVE$^{\mathrm{TM}}$ [5] and console) computer animation (RTICA). For this purpose, we developed new approximations to the Schwarzschild solution of Einstein's field equations in general relativity for light bending around massive bodies. The computational challenge here is that these solutions must be sufficiently efficient for the virtual reality contexts, yet must also remain sufficiently accurate for scientific purposes.

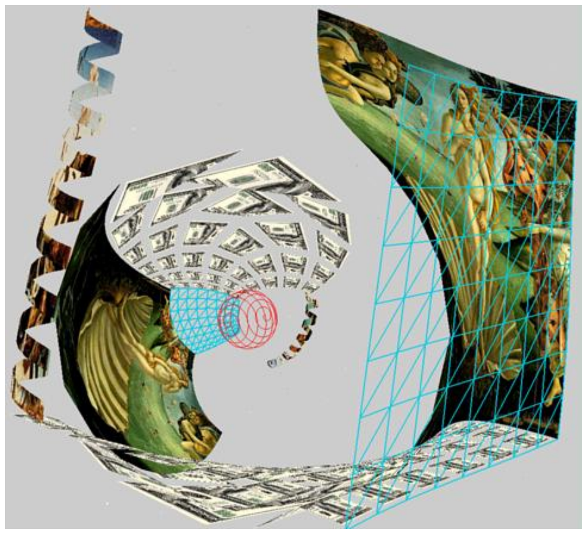

Figure 1: An invisible point mass distorts our view of the world by bending light rays. As second image forms within the Einstein radius about the mass, which is inside the red spiral ball. The rules of this CAVE-to-CAVE game, based on the Prisoner's Dilemma and implemented on the DuoDesk, amused visitors of Alliance'98 and Supercomputing'98.

A prototype, which meets the first, but not the second of these requirements was created by Fullbright exchange graduate student Birgit Bluemer in our illiView team at the NCSA. A salient feature of Bluemer's CAVE application, *graviLens* , is "viewpoint-based calculations," which make it necessary to recompute a deflected position for very vertex making up the scene, for even the slightest motion of the viewer's head.

We incorporated Bluemer's "toy-algorithm" in our *Partnerball* demo on EVL's *DuoDesk* virtual environment, which premiered at the *Alliance'98* conference for the industrial and scientific partners of the NCSA[7]. Here two observers experience their own, differing points of view on the same, simultaneous device, which is an Immersadesk$^{\mathrm{TM}}$ fitted with the DuoView$^{\mathrm{TM}}$ libraries[13] and custom hardware modifications[12]. An invisible "gravitational lens" moves about a scene composed of nearby and faraway objects (stars). Though competing to influence this ball, players must cooperate in determining where it is because each can only determine its direction, not its distance, from their individually perceived distortions of the background. By our rules, player two scores if player one hits the ball with his shooter. If both hit the lens simultaneously the game is prolonged, otherwise it expires in a few seconds. This adaptation of the "Prisoner's Dilemma" lent itself well to the corporate spirit of the occasion.

A phenomenologically more accurate approximation was implemented for the second, and last[1] opportunity to demonstrate the DuoDesk. For Supercomputing'98, our *Superball* RTICA ran simultaneously in the CAVE in Urbana, IL and on the DuoDesk in Orlando[8]. Network communication was implemented using the *Virtual Prototyping System*(VPS) protocol [11], originally developed to accomplish the needs of the Caterpillar Virtual Prototyping Group at the NCSA. For development and test bed purposes, single-user console versions of our software is available for Irix, Linux and Windows platforms. The Windows version, *PingBall*, allows a panelized display of up to 6 simultaneous viewers. This feature helps to obtain a better understanding of the lensing effect. A mathematically more accurate approximation, based on finite difference methods applied to the Schwarzschild geodesic equation, was developed using MATLAB. Another solves the pseudo-geodesic equation by novel methods from algebraic geometry. However, neither of these two could be implemented as real-time interactive CAVE applications (RTICA).

In this paper we give a brief overview of the history of the subject of gravitational lensing. The classic derivation of the geodesic equations for the deflection of light by a mass, and Einstein's approximation for the deflection angle is in [14]. A more modern, standard reference for the "lens equation" is in [16]. We hope that our mathematical exposition will enable ambitious readers to develop their own real-time interactive gravitational lensing animations.

Background

While working on the theory of general relativity, Einstein, in 1915, predicted the phenomenon of the deflection of light by a massive body, and gave an approximation for the deflection angle. This was summarized again in 1936 in his well-known Science article. It appears Einstein thought the result of little value, given that the deflection of light grazing the Sun would be expected on the order of 1.75 seconds of arc, and thus, he thought, imperceptible.

[1]The hardware difficulties of the DuoDesk could not be solved, and the technology did not survive its prototype.

It is interesting to note that as early as 1704, Newton, in the first edition of his *Opticks*, suggested the possibility that gravity acted upon light. This idea was subsequently pursued in the context of a corpuscular theory of light using Newton's law of gravitation by Cavendish (1784), Laplace(1796), and Soldner(1801), resulting in the derivation of a deflection angle exactly half of what Einstein was to later predict.

In 1919, Eddington and collaborators [6], using photographs of a stellar field taken during a solar eclipse in May and again six months later, confirmed Einstein's prediction within a 20–30% uncertainty, small enough to discount the Newtonian result in favor of Einstein's. Subsequent radio-interferometric methods have verified Einstein's result to within 1% [10, 15].

The derivation of the approximate angle of deflection (the "Einstein angle") of light by a massive body is now a standard inclusion in treatises on general relativity [1, 9, 18]. We shall review the argument below.

In the 20's and 30's, various people, including Eddington, Chwolson, and Einstein suggested the possibility of "lensing" phenomena resulting from gravitational deflection of light. In 1937, Zwicky [19, 20] suggested the possibility of galaxies acting as lensing agents. Gravitational lensing was confirmed in 1979 with the discovery [17] of double images of Source 0957+561.

Geodesic Equation

The theory of gravitational lensing begins with consideration of a single massive body; this both informs the study of more general lensing systems [16, p. 29] and is the case with which we shall be most concerned. Therefore, as appropriate for the consideration of a single massive body, we begin with the *Schwarzschild Solution*. Recall, sections 32 and 36 of [14] for example, that this is the solution obtained from the Einstein free-space field equations by imposing the conditions of spherical symmetry and time independence on the metric. That the second imposition is superfluous follows from *Birkhoff's Theorem* [2]. For the discussion below, it is important to recall that the Schwarzschild solution is sensical only outside the spherical shell, $r_s = 2Gm/c^2$, where G is the universal gravitational constant, m the mass of the lens, and c the speed of light.. This defines the *Schwarzschild radius*, which shall be presumed small compared to the radius of the massive body in what follows. (For the Sun this radius is about 3km, well within its interior.) It is customary in the subject of gravitational lensing to refer to a spherical (non-rotating) mass that has radius sufficiently larger than its Schwarzschild radius as a *point mass*.

The path of light moving in the gravitational field of a massive body lies along the null geodesics of the metric. By spherical symmetry, this path must lie in a 2-dimensional plane (for to move off this plane would privilege one of the plane's half-spaces, contradicting spherical symmetry; this is an example of the pre-socratic *Principle of Sufficient Reason*.)

From the Schwarzschild solution, one obtains the following differential equation for the null geodesics of the metric in polar coordinates (r, θ):

$$\frac{d^2u}{d\theta^2} + u = 3Mu^2 \tag{1}$$

where $u := \frac{1}{r}$, and $M := \frac{Gm}{c^2}$ is half the Schwarzschild radius.

We may obtain an approximate solution to this equation by observing that the right hand side of (1) is small compared to u, for

$$\frac{3Mu^2}{u} = 3Mu = \frac{3M}{r} = \frac{3}{2}\frac{r_s}{r},$$

and we have presumed r_s/r small.

This justifies considering $3Mu^2$ as a small perturbation term in the above differential equation.

Setting $\epsilon := 3M$, we thereof consider the equation

$$u'' + u = \epsilon u^2 \tag{2}$$

for which we seek a solution of the form

$$u = u_0 + \epsilon v + O(\epsilon^2).$$

Substituting into equation (2) gives

$$u_0'' + u_0 + \epsilon v'' + \epsilon v = \epsilon u_0^2 + O(\epsilon^2).$$

Equating the terms of zeroth order in ϵ, yield the harmonic equation, $u_0'' + u_0 = 0$, and its solution

$$u_0 = \frac{1}{R}\cos(\theta) \tag{3}$$

where R is the distance to the closest approach of the geodesic to the mass, and θ measured from that line. Note that, $1/u_0 = r_0 = R\sec(\theta)$, is the motion along a straight line a distance R from the mass, as we would expect of a gravitationally unperturbed solution.

Considering now the terms of first order in ϵ, and using (3), we have

$$v'' + v = \frac{1}{R^2}\cos^2(\theta) = \frac{1}{2R^3}(1 + \cos(2\theta))$$

for which a solution is easily found to be

$$v = \frac{1}{2R^2}(1 - \frac{1}{3}\cos(2\theta)) = \frac{1}{3R^2}(2 - \cos^2(\theta)).$$

Collecting these solutions, we find that, to first order in ϵ ,

$$u = \frac{1}{R}\cos(\theta) + \frac{M}{R^2}(2 - \cos^2(\theta)). \tag{4}$$

We shall refer to this as the *pseudo-geodesic equation*. It should be noted[18, p. 188] that a more careful analysis of the Schwarzschild solution obtains the equation of null geodesics in terms of elliptic integrals, which may then be evaluated to arbitrary order in the parameter Mu.

Einstein's approximation

From the pseudo-geodesic equation, one may easily obtain an approximation to the angle of deflection of light by a massive body, for small deflections, as follows.

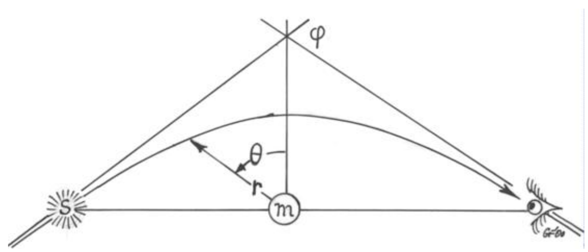

Figure 2: At syzygy the source of light, the deflecting mass, and the observer are (nearly) in a line. The relativistic observer sees the light as Einstein's ring at a deflection angle ϕ.

Consider the path of a photon arriving from infinity and passing a massive object (see Fig 2); we seek an expression for this deflection angle by the angle between the asymptotes. Since the Schwarzschild metric is asymptotically flat, the path of light is asymptotically straight, and we may approximate the deflection angle by difference $\phi \approx \theta(t_{-}\infty) - \theta(t_{\infty})$. At infinity, $u = 0$, and $\theta = \pm(\frac{\pi}{2} + \frac{\phi}{2})$; since ϕ is presumed small, the $\cos^2(\theta)$ term in the pseudo-geodesic equation (4) is negligible. Therefore, in the limit ($r \to \infty$) equation (4) becomes

$$0 = \frac{1}{R}\cos(\frac{\pi}{2} + \frac{\phi}{2}) + \frac{2M}{R^2}$$

i.e.,

$$\frac{2M}{R} = -\cos(\frac{\pi}{2} + \frac{\phi}{2}) = \sin(\frac{\phi}{2}) \approx \frac{\phi}{2},$$

whence Einstein's approximation to the deflection angle

$$\phi \approx \frac{4M}{R}. \tag{5}$$

As mentioned above, this is exactly twice the deflection predicted by Newtonian mechanics under the assumption that light is affected by gravity as would be a material body.

Lens Equation

In studies of gravitational lensing, it is common practice to begin with the Einstein approximation for a point mass and adapt it to the finitistic case by way of the *Lens Equation*. This refers to an interpretation of the approximate deflection angle, ϕ, in terms of the configuration of observer (earth), geodesic deflector (massive galaxy), and light source (distant star), in that order.[2]

Consider the 3 displacement vectors from the observer to the deflector, D_d, from the deflector to the source, D_{ds}, and from the observer to the source, D_s.[3] Thus $D_d + D_{ds} = D_s$. Astronomers can measure the angle ϑ between the deflector and the apparent source (along D_a, the tangent to the light path from source to observer.) But we must rapidly compute an approximation to ϑ just from the relative locations.

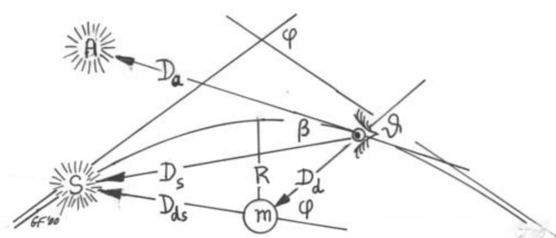

Figure 3: Far from syzygy, classical approximations become inaccurate, as this figure illustrates. To displace a light-source from its true location S to its apparent location A we must find the angle ϑ by fitting a geodesic curve that bends about the mass.

[2]"She could almost pass for thirty-five in the dusk with light behind her." Gilbert and Sullivan.

[3]We use traditional notation for easier recognition.

For sources very far away and almost directly behind the deflector physicists use the approximation $R \approx \vartheta D_d$. Our figure exaggerates the error risked by this and subsequent approximations where vectors are replaced by their magnitudes and small angles equal their sines.

Substituting Einstein's approximation (5) we obtain

$$\phi \approx \frac{4M}{\vartheta D_d}. \tag{6}$$

We derive another relation for ϕ in terms of β, the angle from D_s to D_a (from true to apparent direction of the source.) Applying the law-of-sines to the triangle, but assuming that the $\angle(D_d, D_{ds}) \approx \phi$,

$$\frac{\sin(\vartheta - \beta)}{D_{ds}} \approx \frac{\sin(\pi - \phi)}{D_s} = \frac{\sin(\phi)}{D_s}$$

and dropping sines we obtain a quadratic equation in β:

$$\vartheta - \beta \approx \frac{\phi D_{ds}}{D_s} \approx \frac{4MD_{ds}}{D_s D_d}\,\frac{1}{\vartheta},$$

which produces two values

$$\vartheta_\pm = \frac{1}{2}\beta \pm \frac{1}{2}\sqrt{4\alpha_0^2 + \beta^2}$$

where

$$\alpha_0 := \sqrt{\frac{4MD_{ds}}{D_s D_d}}\,.$$

The two solutions obtained suggest that the source has two images, one on either side of the lens. The angular separation between the two images is then

$$\vartheta_+ - \vartheta_- = \sqrt{4\alpha_0^2 + \beta^2} \geq 2\alpha_0\,.$$

Note that at syzygy, when the observer, mass and source are collinear and so $\beta = 0$, the two observed images lie at $\pm\alpha_0$. It follows from the spherical symmetry of the problem that the whole ring of radius α_0 is the image of the source in this case. Ring-shaped images produced in this manner are known as *Einstein Rings*; the first full Einstein ring has been discovered recently.[3, 4]

The foregoing gives one a clear qualitative picture of gravitational lensing by a point mass. It is equally clear that this approach is unsuitable for the purposes of simulation, as it involves the severe restriction that all angles be small (and makes no statement, even inaccurate, for other angles.)

Toy Lenses

In the development of our lensing code, we have had appeal to several toy lenses; these lenses deform the perceived scene in a manner that is qualitatively similar to that of a gravitational lens. This approach enabled us to display an impressive second image on the DuoDesk and in the CAVE.

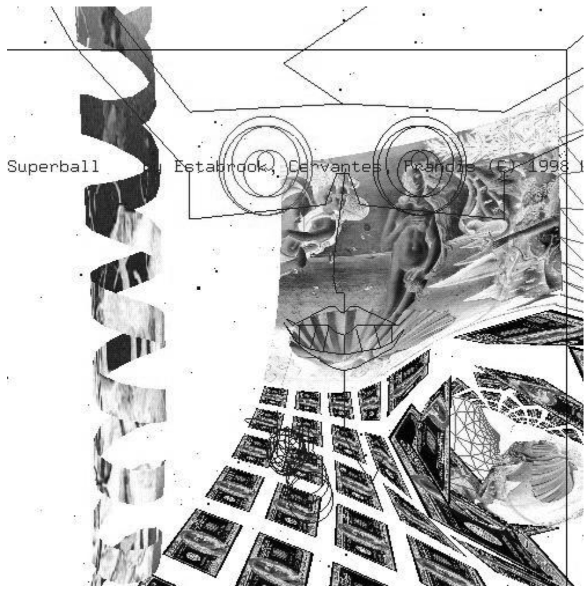

Figure 4: Screenshot negative from *Partnerball* at Supercomputing'98. The avatar for each player consisted of a wire mask (eyes, nose, mouth) tied to the head-tracker, and a wire hand (tied to the wand), with which to launch balls towards the invisible lens. The inner image is visible lower right. Easily recognized distortions of Boticelli's Venus attracted considerable attention.

However, as the above discussion reveals, one really needs a new approach to construct a real-time interactive gravitational lens which avoids accumulating approximations inherent in the Lens Equation formalism.

In a subsequent approach, we worked directly with the pseudo-geodesic equation (4). This equation is related to a family of quartic algebraic curves parameterized by the constant R. Algebro-geometric considerations reveal a solution for the deflection angle in terms of the viewer–source–mass syzygy; standard algorithmic methods in algebraic geometry may be brought to bear upon the problem of finding a practicable expression for the existentially known solution. Implementing such solutions in a fully immersive real-time interactive virtual environment is the subject of our future investigations.

References

[1] Adler, Bazin, and Schiffer. *Introduction to General Relativity*. McGraw-Hill, New York, 1975.

[2] George Birkhoff. *Relativity and Modern Physics*. Cambridge University Press, Cambridge, 1923.

[3] M. W. Brown. Einstein ring caused by space warping found. *New York Times Science Section*, Tuesday 31 March, 1998.

[4] R. Cowen. Gravity's ring: Hubble bags another lens. *Science News*, (April 4), 1998.

[5] Carolina Cruz-Neira, Dan Sandin, and Tom DeFanti. Surround-screen projection-based virtual reality. the design and implementation of the cave. *Computer Graphics (proceedings of SIGGRAPH93)*, pages 135–142, August 1993.

[6] F. W. Dyson, A. S. Eddington, and C. R. Davidson. A determination of the deflection of light by the sun's gravitaional field made during the total eclipse of may 29, 1919. *Mem. Royal Astronomical Society*, (62):291, 1920.

[7] John Estabrook, Ulises Cervantes-Pimentel, Birgit Bluemer, and George Francis. Partnerball. *Alliance'98 Program*, 1998.

[8] John Estabrook, Ulises Cervantes-Pimentel, Birgit Bluemer, and George Francis. Superball. *Supercomputing'98 Program*, 1998.

[9] S. M. Faber. *Differential Geometry and Relativity Theory*. Marcel Dekker, New York, 1983.

[10] E. B. Fomalont and R. A. Sramek. Measurements of the solar gravitational deflection of radio waves in agreement with general relativity. *Physics Review Letters*, (36):1475, 1976.

[11] Volodymyr Kindratenko and Lance Arsenault. Collaborative product design review using distributed virtual reality. *Supercomputing'98 Program*, 1998.

[12] Gary Lindahl. Simulating telecollaborative applications with *duoview*. Technical report, Electronic Visualization Laboratory, University of Illinois at Chicago, 1998. http://www.evl.uic.edu/lindahl/PAPERS/tele.html.

[13] Dave Pape. Cave library version 2.7 design notes. Technical report, Electronic Visualization Laboratory, University of Illinois at Chicago, 1998. http://www.evl.uic.edu/pape/CAVE/prog/Notes.2.7.html.

[14] Yuri Rainich. *Mathematics of Relativity*. Wiley, New York, 1950.

[15] D. S. Robertson, W. E. Carter, and W. H. Dillinger. A new measurement of solar gravitational deflection of radio signals using vlbi. *Nature*, 349:768, 1991.

[16] P. Schneider, J. Ehlers, and E. E. Falco. *Gravitational Lenses*. Springer-Verlag, New York, 1992.

[17] D. Walsh, R. F. Carswell, and R. J. Weyman. 0957+561 A,B: twin quasistellar objects or gravitational lens? *Nature*, 279:381, 1979.

[18] N. Weinberg. *Gravitation and Cosmology*. Wiley, New York, 1972.

[19] F. Zwicky. Nebulae as gravitational lenses. *Physical Reviews*, 51:290, 1937.

[20] F. Zwicky. On the probability on detecting nebulae which act as gravitational lenses. *Physical Reviews*, 51:679, 1937.