Notes on Inversive Geometry

Written by Ethan Bakshy for Math 402, 10AM section, last modified November 5, 2004.

Based on Lecture Notes from F7, October 8, 2004 for Math 402 (Francis)

Further information can be found in sections 2 & 3 of the third supplementary notes, *Chapter 3: The Poincare Models of the Hyperbolic Plane* (http://new.math.uiuc.edu/math402/posteuclid/conformal.pdf)

Inversive geometry is a non-Euclidena geometry relating circles and maps which map circles to circles. Like many of the hyperbolic geometries we study, inversive geometry is conformal, meaning it preserves angles. In inversive geometry, we treat lines and circles the same, so we refer to both lines and circles as *generalized circles*, or sometimes *clines*. Also, infinitey is treated as a single ideal point.

Definition: under the *inversion* of a circle C (with center Q, radius R), every point P in the extended plane goes to another point P'.

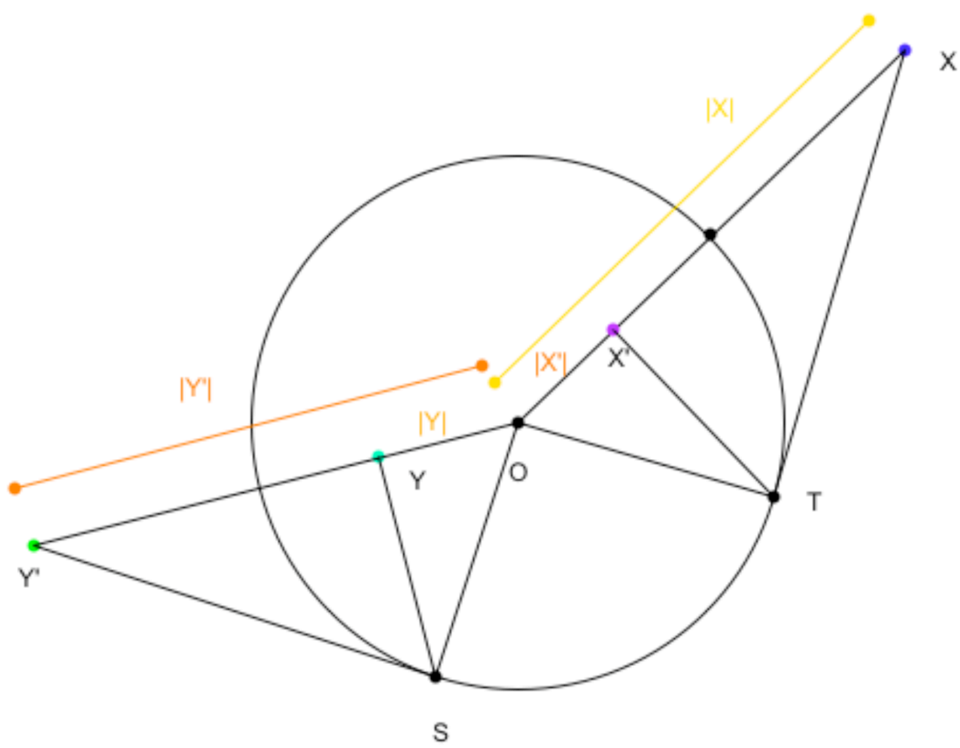

fig. i

Here we see two cases. One in which the point lies outside of the circle (X), whose inverse lies inside the circle, (X'), and another point which lies inside the circle (Y), whose inverse lies outside of the circle. There are two cases which are not depicted in the figure above. When a point lies on the circle itself, it is its own inverse. When the point lies at the center of the circle, its inverse lies at ∞, and vis versa.

We can construct the first case by hand with a compass and a notecard (see fig. i as an example)
(1a) The first case, X lies outside the circle, so X' lies inside the circle. In a way similar to what we have done previously we the Klein Model, we can construct a pole at X. We do this by taking a notecard or right triangle so that one side of the notecard lines up with the center of the circle and the edge of the circle, and the other side lines up with the edge of the circle and X. If we draw a line from X to the point at the corner of the notecard (which we will call T), the triangle formed by the center of the cricle, T, and X is right, and $\overline{\mathrm{TX}}$ is tangent to the circle. If we drop T down perpendicular to $\overline{\mathrm{QX}}$, we get the inverse, X'.

We will examine the first case further with point X when we have a unit circle. The lengths of the radii are 1, so we can derive the relation between the triangles formed by the above construction. Since we have two sides with a length of 1 (the radius), we can say:
$\frac{|X|}{1} = \frac{1}{|X'|}$, therefore $X' = \frac{X}{|X|^2}$.

(1b) We can also do this inversion in GEX. Here is one way to do it. Create a circle (center A), and a point C that is not in the circle. Then, construct a line segment between the center of the circle A and the point C. This is the point we will invert. Now construct a midpoint D of this new line segment, AC. Construct a circle with D as the center point and A as the radius point. Construct the intersection of the two circles. The intersection chosen here is G. We note that the triangle formed the diameter of the new circle, $\overline{\mathrm{AC}}$, and the two line segments $\overline{\mathrm{AG}}$ and $\overline{\mathrm{CG}}$ form a Thale's triangle, so the angle formed by AGC is right, so $\overline{\mathrm{CG}}$ is tangent to our circle centered at A. Then construct a perpendicular, d with G and $\overline{\mathrm{AD}}$. The inverse of C is formed by the intersection of d and $\overline{\mathrm{AD}}$. In the GEX construction, this point was automagically labeled I, but we can also refer to as C'.

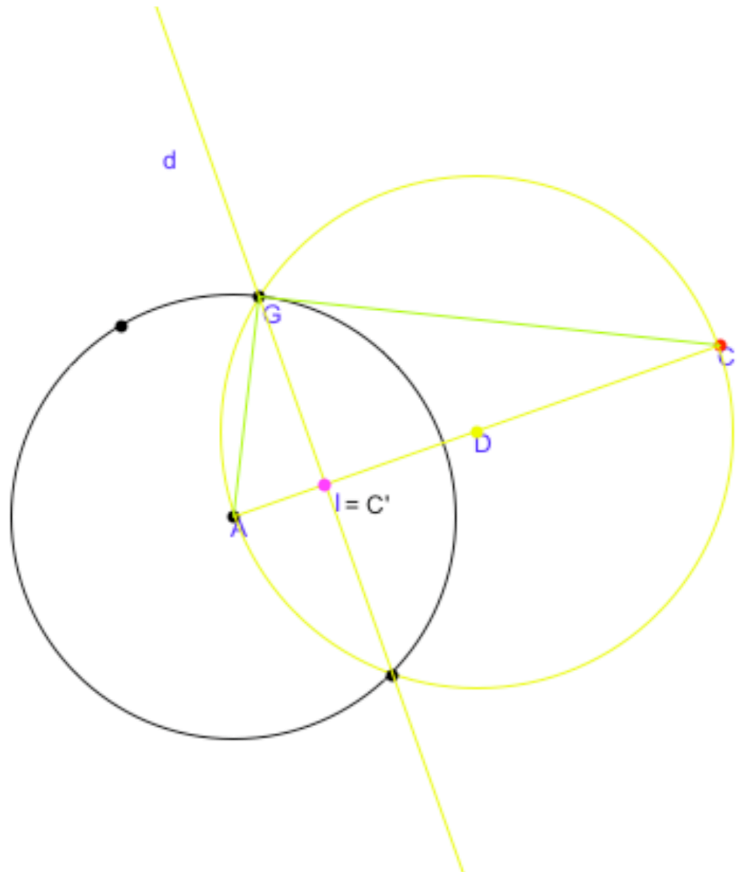

fig ii.a

The following are steps you can use when the point you are inverting is inside the circle . *It is recommended that you pick a point that is not too close to the center, otherwise its inverse might not fit on the page!*

(2a) Here is how we would go about constructing the first case with a compass and a notecard: The second case, Y lies inside the circle, so Y' lies inside the circle. Draw a line starting at the center (O) of the circle that goes thru Y and extends out of the circle. Then construct a line perpendicular to $\overline{OY}$, from Y to the edge of the circle. We can label this point S. Then, with your notecard, line up one side with the center of the circle and S. The inverse of Y, Y', is at the intersection of your notecard and the line that was drawn thru the center of the circle and Y. If it does not intersect, make sure that line was long enough!

(2b) We can also construct the inverse of a point inside a circle in GEX. Create a circle, and label the center A. Then create a point inside this circle that we want to invert, which we will call C. Now create a ray from A thru C. The inverse, C' will lie on this ray. Construct a line perpendicular to AC at C, and contruct an intersection of that perpendicular and the circle. Here, we label the intersection D. Construct a line segment from A to D. Then construct a perpendicular to $\overline{AD}$ at D. Construct the intersection of that perpendicular and the ray $\overrightarrow{AC}$. Here, GEX labels this intersection as F, which is C', the inverse of C. Note that the $\overline{DF}$ is tangent to our circle, because the angle ADF is right, since $\overline{DF}$ is perpendicular to $\overline{AD}$.

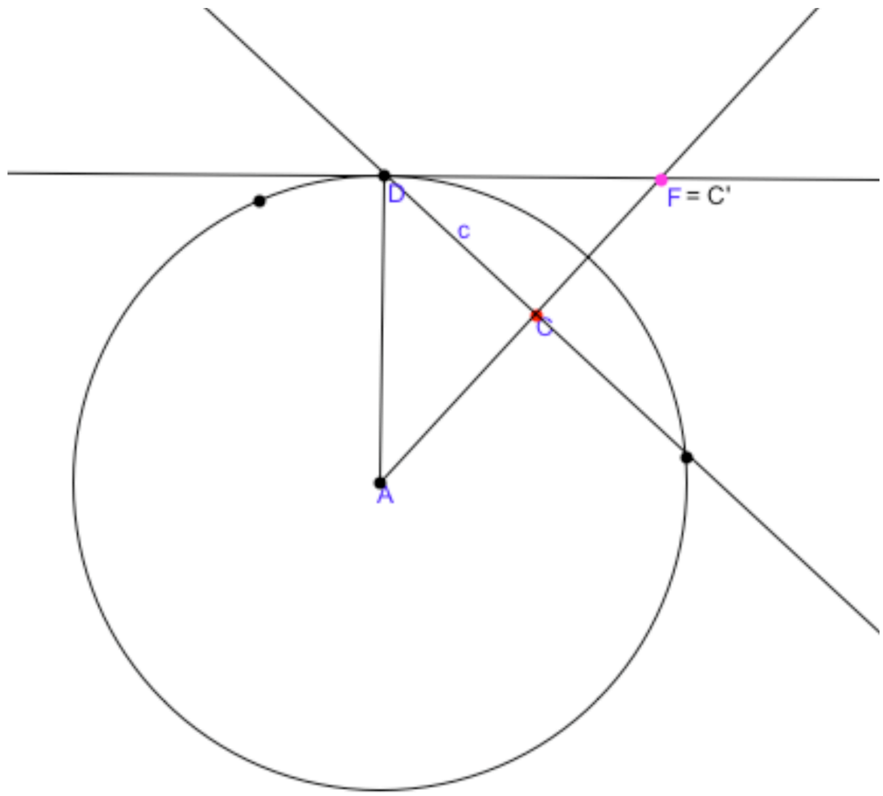

fig ii.b

Exercise 1: given a circle (Q, R) and some point X inside the circle, what is Y (the inverse of X)?

Hint: Y-Q = displacement vector if Q is the origin

$$Y - Q = r^2 \frac{X-Q}{|X-Q|^2} \text{ where } r = |Q - R|$$

Properties of Inversion

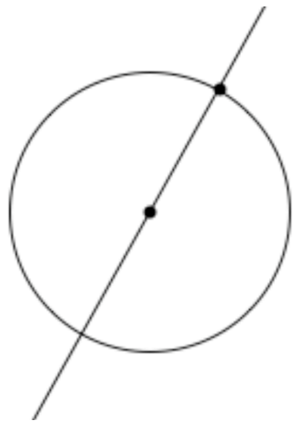

fig. iii

Exercise 2: What is the inversion of a line going thru the center (fig iii)? They invert themselves. Prove this.

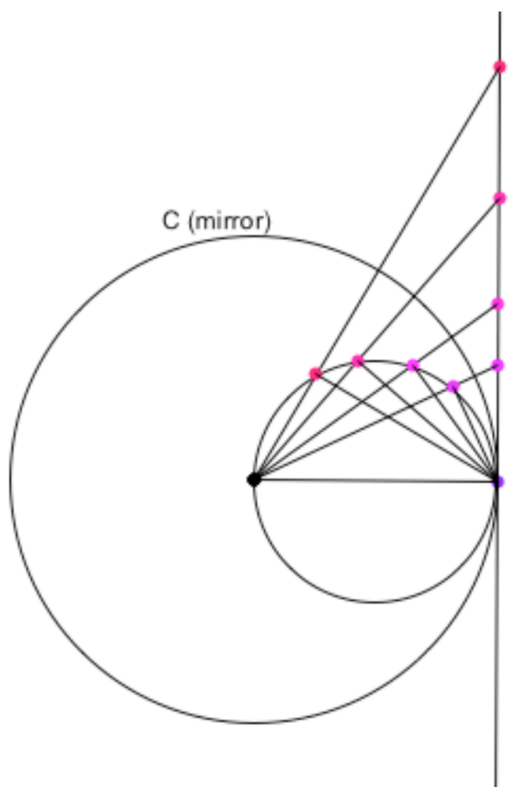

fig. iv

Exercise 3: tangent lines invert to circles tangent to lines (and thru the center). (fig iv) Prove this

Definition: Lines are extended circles thru ∞ (an infinitely large diameter)
Theorem: inversion maps circles to circles

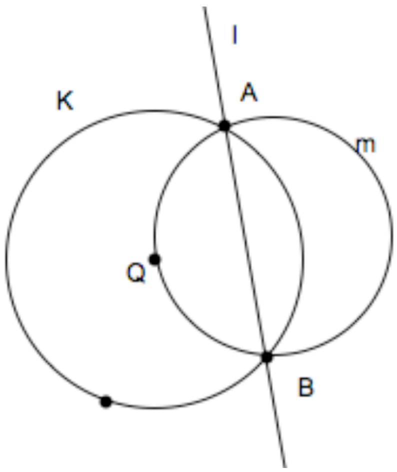

fig v

Lines thru K not thru center (Q) invert to circles thru the center and thru two radial points, A, B where l crosses K.
Proof:
$\infty \in l \Rightarrow Q \in m$
$A, B \in K \Rightarrow A = A$ and $B' = B'$
then m is in a circle thru Q, A, B.

Excercise 4: What is the inverse of a line extending a proper chord? (see fig v)

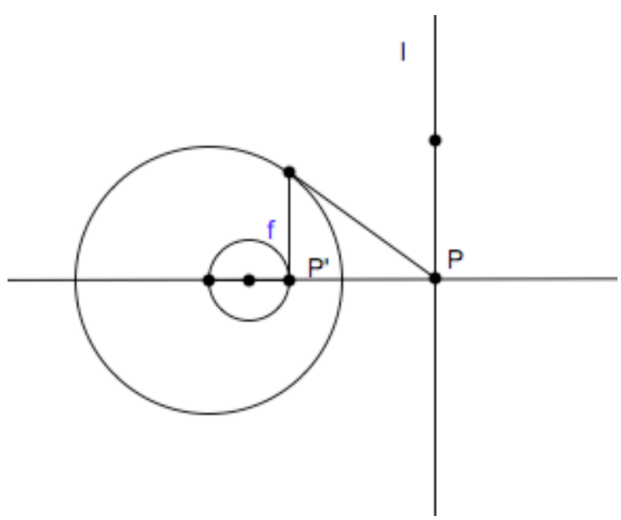

figure vi

Exercise 5: What is the inverse of a line not touching a mirror? (fig vi)