# Lesson on Matrices and then Some

GF, for Math198 on M6

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## 1 Introduction

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \hdotsfor{5} \\ x_{d1} & x_{d2} & x_{d3} & \dots & x_{dn} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{d1} & x_{d2} & x_{d3} & \dots & x_{dn} \end{bmatrix}$$

Here is a (partial) writeup of what I said at the whiteboard on M6. It is also a LaTeX document, and hence a useful example of how to write it. For example, this first image is a screen print of the code that composed this matrix. If you wish to see its LaTeX code, open matricesM6.tex in a text processor.

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \hdotsfor{5} \\ x_{d1} & x_{d2} & x_{d3} & \dots & x_{dn} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{d1} & x_{d2} & x_{d3} & \dots & x_{dn} \end{bmatrix}$$

Note that the subscripting begins the natural numbers at 1. We shall get used to counting from 0 and ending one short of how many items we enumerate. Also note that the author has a mnemonic name for the final index pair, ”d” for ”down” but ”n” for the horizontal extent.

I mentioned that physics uses sub and superscripts. But for most of you this is not memorable at the moment. But we do think of points and vectore in 2, 3 and 4 space as columns. Hence a matrix can be considered (horizontal) vector of column vectors. However you may have learned to multiply matrices here is an equivalent way which has more geometrical meaning.

Finally note another LaTeX thing you'll have to get used. First, LaTeX has its own typesetting rules where to put images relative to the text. Second, we (absentmidedly) make a mistake above (only one? I hope). We wrote ”d” instead of “d”. See the difference. The begin quotation symbol is made with two back-tics (upper left on your keyboard). Get used to that too.

## 2 Matrix times Vector

We interpret the product

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & g \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} x + \begin{bmatrix} d \\ e \\ f \end{bmatrix} y + \begin{bmatrix} g \\ h \\ k \end{bmatrix} z = \begin{bmatrix} ax + dy + gz \\ bx + ey + hz \\ cx + fy + kz \end{bmatrix}$$

with the same result (rightmost) you would expect from highschool.

Of course, we don't like to burden our typing with a lot of extra letters that serve no particular purpose. So we might write the the matrix as a row-vector and, if necessary, decorate the letters that represent entire column-vectors some conventional way, or not at all (being mathematicians, apprentice or otherwise.) Thus

$$\begin{bmatrix} \overrightarrow{k_0} & \underline{k_1} & K_2 \end{bmatrix} \underline{x} = \underline{k_0} x_0 + \underline{k_1} x_1 + \underline{k_2} x_2$$

Which decoration do think I like best? Note also that if $\underline{x}$ denotes a vector, it is natural to name its components $x_0, x_1, x_2$.

What does this mean? Think of the columns of the matrix as the coordinats of three points in space, and also for the vector pointing from the origin to that point. (This is a common double meaning in analytic geometry.) So it is a coordinate system somebody has sat on and deformed from the usual unit and mutually perpendicular coordinate axes. The coordinates of the point being *transformed* by the matrix is the point at the end of the *linear combination*, namely the sum of product of the axes vectors by the coordinates of the point $x$. We'll keep loosing that underline decoration for ease of typing, if not for ease of reading. Get used to inferring the context.

The technical term for this is an *axonometric transformation* and then we can distinguish the *axons*, which are vectors, from the *axes*, which are infinite lines through the vectors.

Note that I have just introduced you to the custom of emphasizing technical words, also called *jargon*, using the italic font. It written "escape emph..." in the .tex file. We can reserve quotes for what they are historically meant for. If you're writing by hand, and can't really write italics, use the convention of underlining what is meant to appear in italics. So ... no more "air quotes". (This is correct, because air quotes aren't part of English orthography yet, so we use quotes to mean an ironic or metaphoric use of the phrase inside them.)

## 3 Product of Two Matrices

We next apply a common mathematical method of *generalizing*, in this case that of matrix multiplication. Suppose we want to multiply two matrices in the *same way* that we multiplied a matrix and a vector. Thus given two matrices $M, P = [p_0, p_1, p_2]$ we define

$$MP = [Mp_0, Mp_1, Mp_2]$$

. Thus the axons (column vectors) of the product $MP$ consists of the M-transformed the axons of $P$.

Since multiplication of numbers is associative, so too shall matrix multiplication be. And we can write

$$MP\underline{x} = (MP)\underline{x} = M(P\underline{x}).$$

You already know what the first association means (but explain it when you rewrite this lesson in you Journal in your own words, with examples.)

The second shows that matrix multiplication represents composition of two transformations, one after the other. (Careful, we read "after" from right to left, like functional composition, not like spelling a word.) The third is why all of this is important in computer graphics (and math, of course.)

Suppose moment the axons in both $M, P$ are independent, even two sets of *orthogonal frames*, a.k.a mutually perpendicular unit vectors. The we can read the composition $MPx$ as *placing* the coordinates of $\underline{x}$ not into the usual coordinate system, but in the axonometric coordinate system given by the columns of $P$. Then we place those into the axonometric coordinate system given by the columns of $M$.

So a matrix also has two meaning (like the coordinates of a point and its vector). As a transformation that *moves* every point in space to a new location. For instance, a rotation. Or, as the axons of a new coordinate system. One likes to call the former the *alibi* approach: the culprit has changed places. The latter becomes the *alias* approach, the culprit has new names depending on which coordinates you use.

**Exercise 1.** Give a change from a Cartesian (rectangular) coordinate system in the plane to polar coordinates a alibi-interpretation!

**Exercise 2.** Give the composition of two rotations $MP$ an alias-interpretation.

**Exercise 3.** Put the solution of these exercises into your Journal.

**Exercise 4.**Then, someday later (but not too much later), type this into LaTeX as an exercise in writing LaTeX. Obviously you can copy code from this document, shamelessly. Put this into your repository in a subfolder name LaTeX.

# 4 Omissions

I added a lot of detail here that I surely did not say, however hurriedly on Monday. But I also left some stuff out. Perhaps you can fill that in from your class notes and memory? It was about *translations* not being represented by matrix multiplication. But if you add on dimension and use homogeneous coordinates then even translation becomes a matrix multiplication. That is how computer graphics libraries work. But that must have its own lesson someday soon.

Also recall the body language that showed that products of two rotations are rarely commutative. (Think, when would they be so?)