A GENERALIZED WAGON WHEEL EFFECT ON SMOOTH TORUS-VALUED FIELDS RESTRICTED TO $\mathbb{Z}$-LATTICES

AARON WITTRIG

Abstract. When looking at a wagon wheel (or a spoked car wheel, fan blade, propellor, etc.) undergoing radial acceleration, one often gets the effect that the spokes are stationary, moving slower than they should, or beginning to move backwards. These effects are especially obvious under stroboscopic conditions (e.g. in a film or under a strobe light), in which the apparent dynamics of each spoke can be given by restricting its actual position function ($\mathbb{R} \to \mathbb{T}$) to the set of exposure times, some contiguous subset of $\mathbb{Z}$. In this short paper, we explore the dynamics of these restrictions in more generality, in the hope that we may foster further research in the field of wagon wheel stroboscopy.

1. Introduction

For the sake of concreteness, we will begin with a (somewhat) physically plausible example. Imagine a wagon wheel with a single spoke undergoing a constant radial

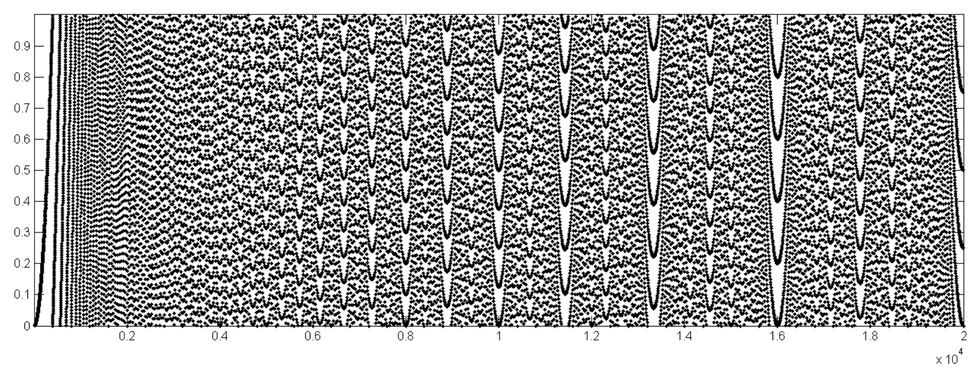

Figure 1. $\{(t, \theta(t)) : t \in \{0, ..., 20000\}\}$

acceleration of 2 revolutions per second per second, starting from rest. Facing the wagon wheel is a camera which records the position of the spoke every five-hundredth of a second. Figure 1 shows a plot of the $\{(t, \theta(t))$ for $t \in \{0...20,000\}\}$, where $\theta(t) = \left(\frac{t}{400}\right)^2 \mod 1$ is the angle (as a fraction of 2π) the spoke at the instant of the t'th camera exposure makes with its initial position. Notice the parabolas that emerge

Figure 2 shows $\{(t, \theta(t))$ for $t \in \{0, ..., 40000\}\}$ (for this and future plots, we shall supress the axes to make room). For reasons which shall be ellucidated shortly, we can determine the entire sequence $(\theta(t))_{t\in\mathbb{Z}}$ from the values it takes on these integers, as $\theta(40000 + t) = \theta(40000 - t)$ and $\theta(80000 + t) = \theta(t)$. In fact, for any $p(t) \in \mathbb{Q}[t]$, the sequence $(p(t) \mod 1)_{t\in\mathbb{Z}}$ is periodic.

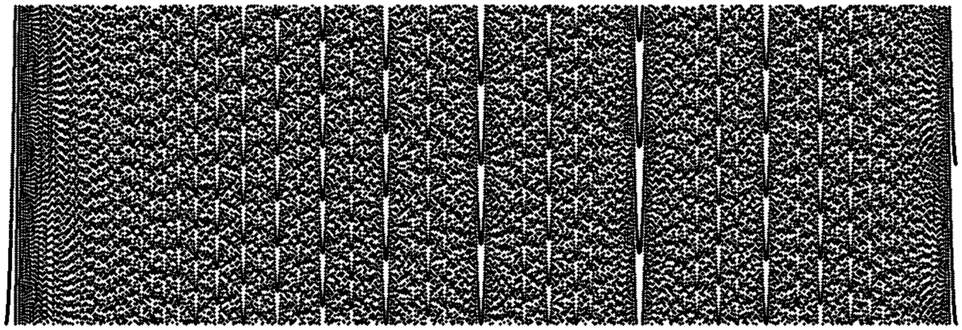

FIGURE 2. $\{(t, \theta(t)) : t \in \{0, ..., 40000\}\}$

Doubling the capture rate to 800 fps, we get a similar sequence as before, though more elaborate. Figure 3 shows $\{(t, \phi(t))$ for $\phi(t) = \left(\frac{t}{800}\right)^2 \mod 1$ and for $t \in \{125000, ..., 195000\}\}$

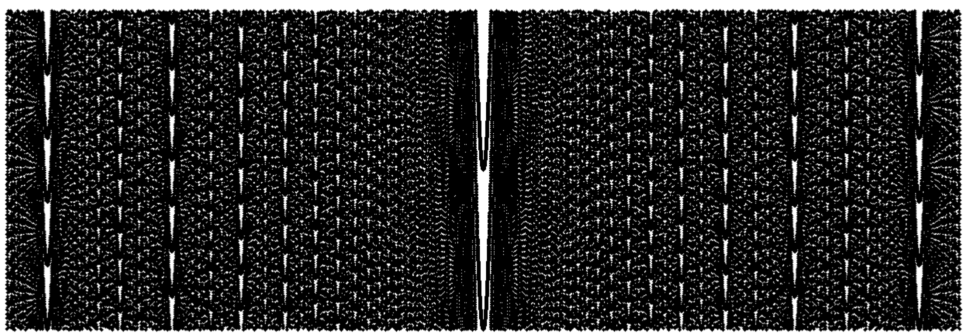

FIGURE 3. $\{(t, \phi(t)) : t \in \{125000, ..., 195000\}\}$

Notice the cascades of parabolas apparent in figures 1-3. Their appearance is not unique to quadratic functions, or even polynomials. Figure 4 shows a plot of $\{(t, \psi(t))$ for $\psi(t) = e^{t/10^7} \mod 1$ and for $t \in \{182911016, ..., 182982128\}\}$ Unlike rational polynomials mod 1, the sequence $(\psi(t))_{t \in \mathbb{Z}}$ is aperiodic, and admits higher order structure later in the sequence. Figures 5-16 exemplify this structure.

In what follows, we will derive a taylor like approximation theorem which will describe these structures and why they occur.

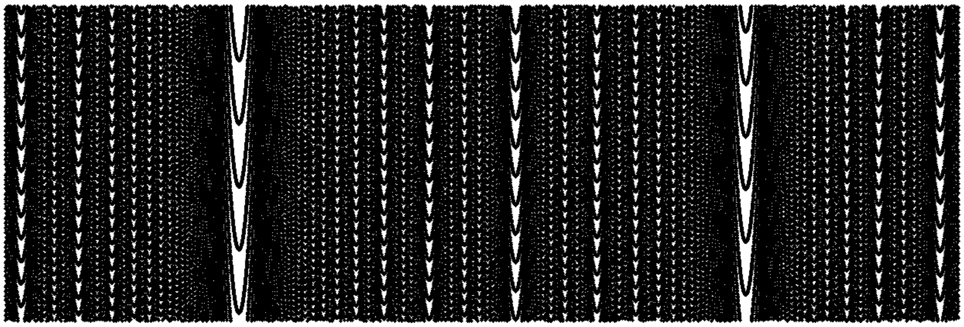

FIGURE 4. $\{(t, \psi(t)) : t \in \{182911016, ..., 182982128\}\}$

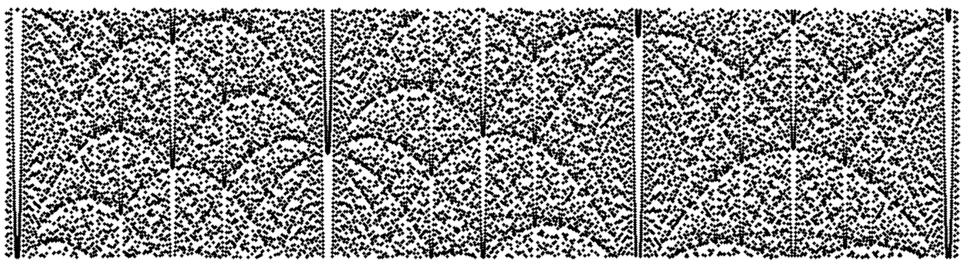

FIGURE 5. $\{(t, \psi(t)) : t \in \{237089295, ..., 237104777\}\}$

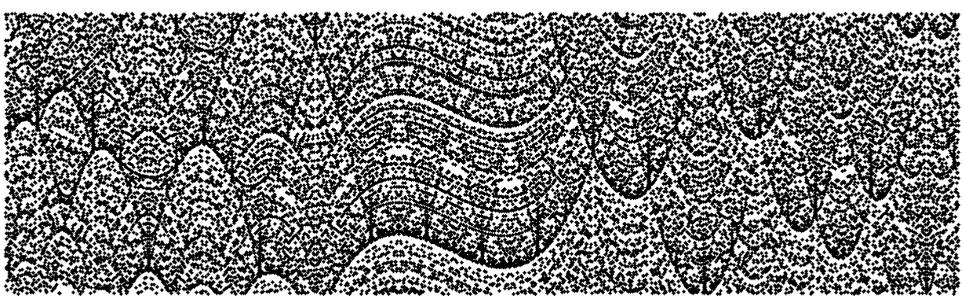

FIGURE 6. $\{(t, \psi(t)) : t \in \{248086668, ..., 248103955\}\}$

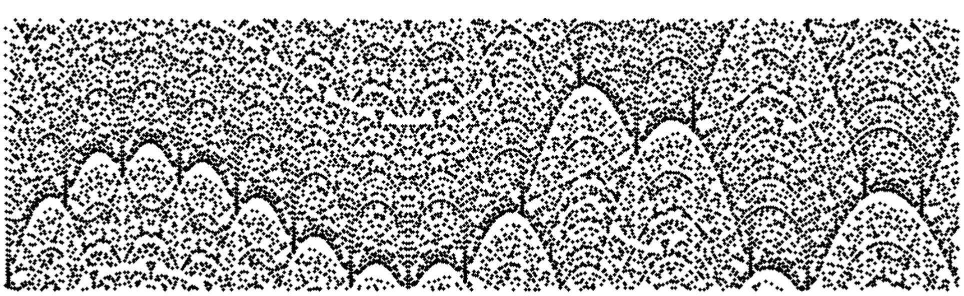

FIGURE 7. $\{(t, \psi(t)) : t \in \{255017933, ..., 255035394\}\}$

Figure 8. $\{(t, \psi(t)) : t \in \{256552147, ..., 25656417\}\}$

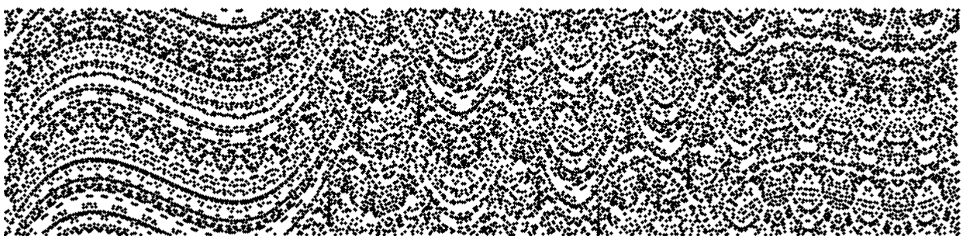

Figure 9. $\{(t, \psi(t)) : t \in \{260616695, ..., 260631194\}\}$

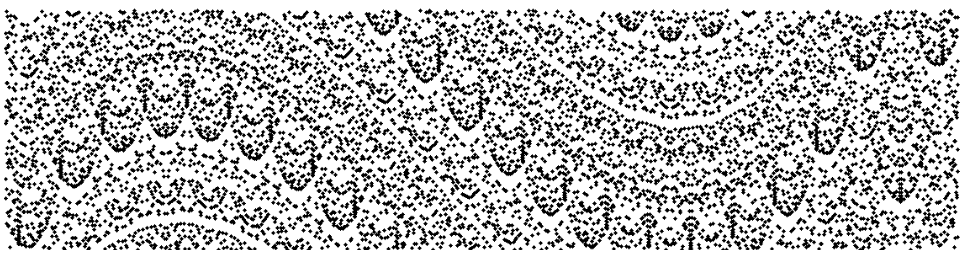

Figure 10. $\{(t, \psi(t)) : t \in \{264217235, ..., 264224830\}\}$

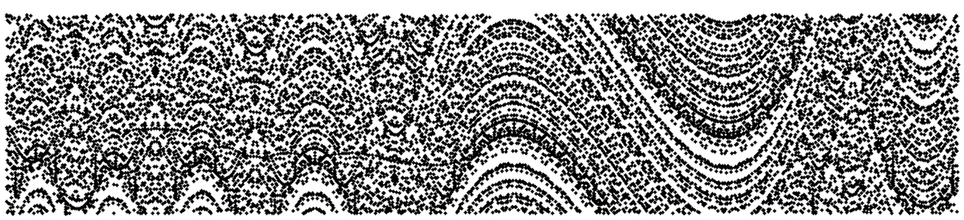

Figure 11. $\{(t, \psi(t)) : t \in \{264833067, ..., 264844570\}\}$

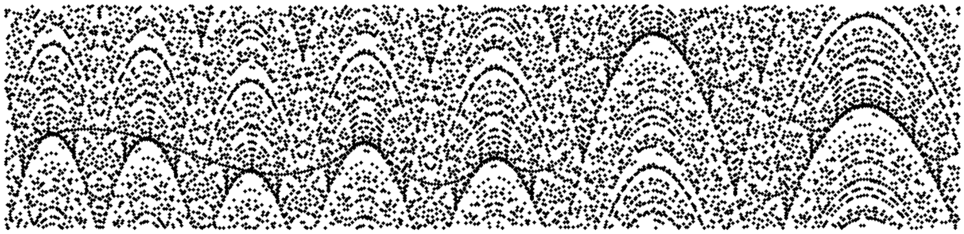

Figure 12. $\{(t,\psi(t)) : t \in \{267543524, ..., 267563524\}\}$

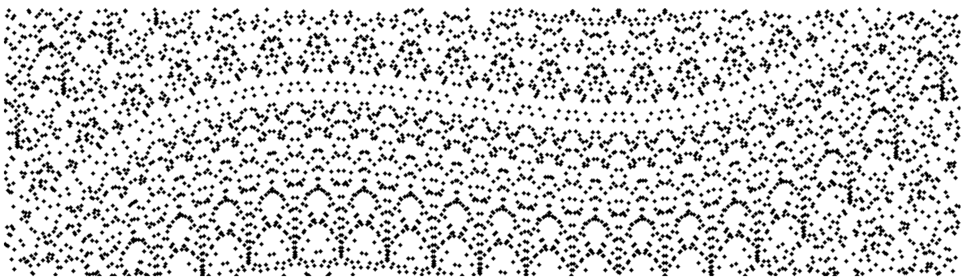

Figure 13. $\{(t,\psi(t)) : t \in \{268154289, ..., 268158951\}\}$

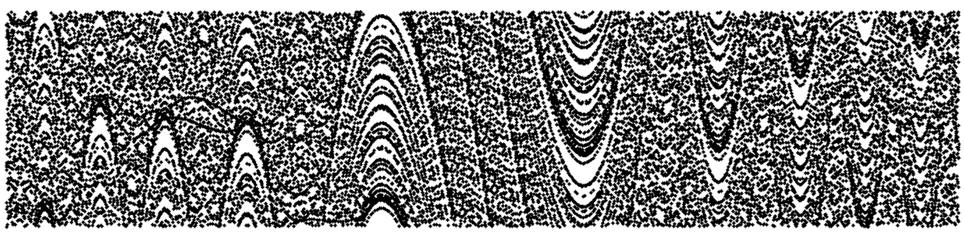

Figure 14. $\{(t,\psi(t)) : t \in \{271112273, ..., 271132273\}\}$

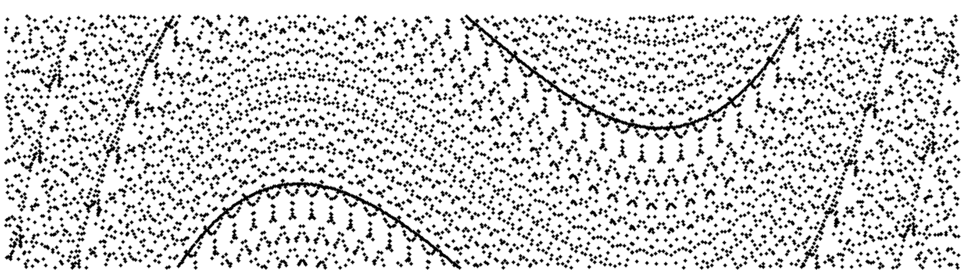

Figure 15. $\{(t,\psi(t)) : t \in \{274401059, ..., 274406997\}\}$

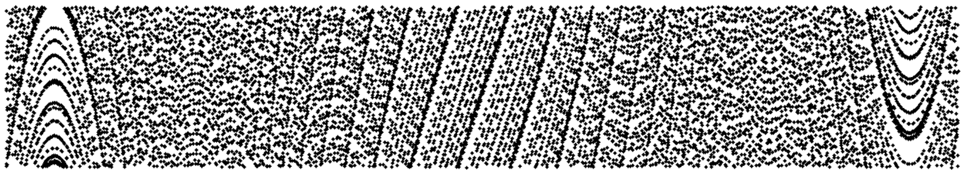

Figure 16. $\{(t, \psi(t)) : t \in \{283645153, ..., 283654653\}\}$