Math Visualizations
Billiard Trajectories

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1 Billiard Trajectories

1.1 Introduction

Suppose we have a squared billiard board and at each corner there is a hole with shape a quarter of a circle (with radius ϵ). If we shoot a tiny ball or particle (it is just one point and does not have size) from one fixed corner at a certain angle θ and suppose the ball do not lose any speed until it falls into a hole, then an interesting question is whether the ball is guaranteed to fall into a hole finally. Moreover, which holes will capture it more frequently if we change the value for angle θ? What are the lengths of the trajectories?

1.2 Background

In fact, it is guaranteed the ball will fall into one of the other three holes (distinct from the one at the corner where it was shot from) after a finite time. Moreover, each hole will have probability $\frac{1}{3}$ to capture the ball. The length of the trajectory is worth studying and Prof.F. P. Boca, Prof.R. N. Gologan, and Prof. A. Zaharescu have papers containing results about the average length of the trajectories (over all possible angle $\theta \in [0, \frac{\pi}{2}]$). By the property of reflections, studying the trajectory on the squared board is the same as studying the trajectory on the corresponding torus if we glue the opposite sides of the square accordingly. Moreover, we could consider the universal covering space of the torus. In this case, the four cornered holes will become a full circle at each lattice point on $\mathbb{R}^2$ and the trajectory will just become a ray on $\mathbb{R}^2$. The first hole the ball hits is just the first circle on any lattice point the ray hits first. We will prefer to make the radius ϵ small and study the limit, but another interesting version of this problem is studying the situation where ϵ is changing while the ball is running.

1.3 Goals

(1) Draw the squared billiard board and a grid board. Let the trajectory on the billiard board coincide with the ray at the grid board.

(2) Draw the trajectory on a torus and make all three trajectories coincide and grow together.

(3) Allow the user to choose the values of θ and ϵ.

(4) Output the trajectory length dynamically.

(5) Finally, if possible, allow the user to type in a function of an initial value ϵ_0 and make ϵ change accordingly.

1.4 Tools

- Use HTML Canvas and Javascript to implement the graphics.
- Use Three.js to help draw the torus.
- Use correct mathematical parametrizations to make those trajectories coincide and increase simultaneously.

2 Another Possible Future Project: Beatty Sequence

(a) Single Beatty Sequence:

Create a number line. Let the users choose the value for α. Users can choose to see the whole B_α altogether, or see it element by element. In the latter case, we allow the user to click a button and each time they click, the next number in B_α will be marked as a bigger rod dot on the number line.

(b) Beatty Partition Let the users be able to add multiple number lines, so they can compare different beatty sequences if they want.