Quadrics*

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January 20, 2008

First draft W1

In 3D, this

$$ax^2 + by^2 + cz^2 + dxy + eyz + fzx + gx + hy + kz + m = 0$$

is the formula for a *quadric surface*. The quadric surface is the locus of points (x, y, z) that satisfy the equation. The 10 parameters $a, b, c, ..., m$ determine the shape of the quadric. We would like to know how they do this.

In 1D, the familiar *quadratic equation* $ax^2 + bx + c = 0$ was studied to death in high school. But we're not quite done with it. Here it will serve us as the bottom rung of the *dialectic*, a.k.a. *dimensional ladder*, that will enable us to peek into the 10D paramater space for quadrics.

Unfortunately, in 1D the entire solution is described by the familiar

$$ax^2 + b^2 + c = 0 \text{ if and only if } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which consists of a single point when $b^2 = 4ac$, and has two or no (real) solutions, depending on $b^2 <$ or $> 4ac$. Why don't you graph the critical surface in 3-dimensional a, b, c space using DPGraph. This surface, $b^2 = 4ac$, is called the *discriminant* of the quadratic equation, and for (a, b, c) on the discriminatn surface, there is only one solution. On one or the other sides of the surface there are none, or two solutions.

Solution: DPGraph shows $y^2 = 4xz$ very well, but because the axis of this double cone is not aligned with the xyz-axes, it is easily mistaken for some other quadric. If we make the *substitution*

$$\begin{aligned} x &\leftarrow \frac{x - z}{2} \\ y &\leftarrow \frac{x + z}{2} \\ z &\leftarrow z \end{aligned}$$

*For MA198 Spring 2008. University of Illinois.

we get the more recognizable equation

$$y^2 = x^2 - z^2 \text{ or } x^2 + y^2 = z^2$$

whose locus has horizontal (x, y) sections circles of radius $|z|$. For $x = 0$ we factor

$$0 = z^2 - y^2 = (z - y)(z + y)$$

for the equation of two crossing lines. Thus the discriminant surface is a double cone.

Now, lets (try) to do the same thing to 2D. Note that the quadratic equation is just a (somewhat very!) special case of the equation for a quadric surface, namely $b = c = c = e = f = h = k = 0$. The loci of the 2D quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

are called the *conics* because these curves are all obtained by cutting a (double) cone in 3D by a plane (you learned that in high school, right?)

One approach to solving quadratic equations is very geometrical (as will be apparent much later.) It consists of turning the quadratic equation into a *homogeneous* equation. We do this by first setting a new variable $w = 1$ and then mutliplying through by 1 a number of times.

Let's see how this works in 1D. We write $ax^2 + bxw + cw^2 = 0$. Note that this is a special case for the 2D quadratic equation ($d = e = f = 0$), provided we release $w = 1$ and let it be 2nd variable in 2D xw-space. (Our letters may vary from paragraph to paragraph, but not their roles. Letters at the end of the alphabet are spatial coordinates, and the parameters are at the beginning of the alphabet.)

We now solve the 1D homogogenous quadratic equation as follows.

If $a = 0$ and $b \neq 0$ (if it were, we'd have nothing to look at, right?) then the LHS factors into $w(bx + cw) = 0$, which is true if $w = 0$ or if $(bx + cw) = 0$, both of which are equations of straight lines in xw-space. To get back to 1D, we set $w = 1$ (which is still another line) and look at the intersection the locus (loci) makes with the special line $w = 1$.

If $a \neq 0$ then we can eliminate it by dividing through the equation. Note that the RHS $0/a = 0$ doesn't change, so the locus can't change either when we solve

$$
\begin{aligned}
x^2 + 2\frac{b}{2a}xw + \frac{c}{a}w^2 &= 0 \\
x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}w^2 &= (\frac{b^2}{4a^2} - \frac{c}{a})w^2 \\
(x + \frac{b}{2a})^2 &= (\frac{\sqrt{b^2-4ac}}{2a}w)^2 \\
(x + (\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a})w)(x + (\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a})w) &= 0
\end{aligned}
$$

The last line is the equation of two lines crossing at the origin. When we set $w = 1$ again, these two lines cross this horizontal at the two solutions to the quadratic equations.

The foregoing was an example of how to solve an inhomogenous equation in some dimensions, by solving a homogeneous equation in one dimension higher. Working in homogenous coordinates is an essentail aspect of computer graphics as it has been in algebraic geometry for hundreds of years.

Note that by solving the 1D quadratic equation in the foregoing way, also solved the special case of the 2D quadratic equations $x^2 + bxy + cy^2 = 0$, only with different letters. We next consider the "latter half" of the entire quadratic equation, namely the case of *linear equations* in the plane, $dx + ey + f = 0$, except that we rewrite this equation again, as $ax + by + c = 0$. This time we want to know how to characterize each line such an equation defines. In high school you knew lines by their "point-slopes", or by two points the line passes through, or some other geometrical property of the line. Now we want to understand the lines in the plane as the *points* in some space, which we shall call the *moduli space* of lines in the plane. More familiar names for this might be *configuration space*. In the present case, it is not the same as the *parameter space*, which is the 3D *abc*-space. Why? Because the point (a, b, c) is not unique to the line. The equation $tax + tby + tc = 0$ describes the exact same line, provided that $t \neq 0$, of course. (Think: what is the locus when $t = 0$.) The Greeks would have said that it isn't the triple (a, b, c) that determines the line, but their ratio $a : b : c$. So, now we want to know how to imagine the space of all ratios $a : b : c$, except $a = b = c = 0$.

$$
\pounds = \{a : b : c | 0 \neq a^2 + b^2 + c^2 = r^2\} \text{ and } a : b : c = \{(ta, tb, tc) | t \neq 0\}
$$

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To solve this visualization problem we shall need some topology. First off, we chose a special value, $t = 1/\sqrt{a^2 + b^2 + c^2}$, so that $(a/r, b/r, c/r)$ is where the line pierces the unit sphere, written S^2. Unfortunately, we still have two points specifying the line, because $(-a/r, -b/r, -c/r)$ is a point on the unit sphere in *abc*-space for the same line in the *xy*-plane. These two points are *antipodes* because they are opposite each other through the center of the sphere.

In some circles of topology people would content themselves by saying that the moduli space, P^2, we are seeking to visualize is the sphere with antipodes indentified to single points,

$$P^2 = S^2/(a, b, c) \sim (-a, -b, -c)$$

. But this is too hard to visualized. So we proceed by throwing away one of each pair of antipodes, leaving only one equatorial hemisphere. Each of the points strictly below the equator correspond to a unique line in the $xy-plane$. But on the equator we still need to identify antipodes. We could throw away half the equator, and we would have a picture of the moduli space. But it have a very "raw edge" on the equator. Sow we proceed to try and suture this raw edge. While there is a nice visualization of this procedure, it's not the easiest to appreciate without computer graphics[1]. So we back up a little.

Start with the sphere and decompose it into three pieces, a belt along the equator that extends the same distance above as below the equator. What remains are two disjoint polar caps. Every point in one cap has its antipode in the other. So we discard one of the caps and keep the other. Just remember we have to sew it back to the equatorial belt eventually.

The belt also contains antipodes. This time we cut it in half, leaving an "east strip" and and "west strip". Again each point in one strip has its antipode in the other (and vice versa), so we discard one of them. We're not done. The vertical (short) cuts on the west strip are antipodes. But now we can identify these, provided we put a half-twist into the strip. We get a *Moebius strip.* This has a single circle for an edge, to which we propose to glue the remaining cap. Of course we can't do that in 3-space.

If we contort the Moebius strip and the cap just right, and allow the surface to pass through itself, then we can, and so obtain what is known as a *Boy surface*, because Werner Boy first figured out how to do that.

Now we are done. We have proved that the moduli space for the lines in the plane is a Boy surface. For entirely different reasons, it is also called the *projective plane.* Visualizing a Boy surface is visualization problem left for another time.

[1]See the illiSnail real-time interactive compuer animation.