Quadrics*

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First draft WI

In 3D, this
$$ax^2 + by^2 + cz^2 + dxy + eyz + fzx + gx + hy + kz + m = 0$$
is the formula for a *quadric surface*. The quadric surface is the locus of points (x, y, z) that satisfy the equation. The 10 parameters $a, b, c, ..., m$ determine the shape of the quadric. We would like to know how they do this.

In 1D, the familiar *quadratic equation* $ax^2 + b^2 + c = 0$ was studied to death in high school. But we're not quite done with it. Here it will serve us as the bottom rung of the *dialectic*, a.k.a. *dimensional ladder*, that will enable us to peek into the 10D paramater space for quadrics.

Unfortunately, in 1D the entire solution is described by the familiar

$$ax^2 + b^2 + c = 0 \text{if and only if} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which consists of a single point when $b^2 = 4ac$, and has two or no (real) solutions, depending on $b^2 <$ or $> 4ac$. Why don't you graph the critical surface in 3-dimensional a, b, c space using DPGraph. This surface is called the *discriminant* of the quadratic equation, and for (a, b, c) on the discriminatn surface, there is only one solution. On one or the other sides of the surface there are none, or two solutions.

Solution: DPGraph shows $y^2 = xz$ very well, but because the axis of this double cone is not aligned with the axes, it is easily mistaken for some other quadric. If we make the *substitution* $x \leftarrow x - z, y \leftarrow x + z$ we get the more recognizable equation

$$y^2 = x^2 - z^2 \text{ or } x^2 + y^2 = z^2$$

whose locus has horizontal (x, y) sections circles of radius $|z|$. For $x = 0$ we factor

$$0 = z^2 - y^2 = (z - y)(z + y)$$

for the equation of two crossing lines. Thus the discriminant surface is a double cone.

Now, lets (try) to do the same thing to 2D. Note that the quadratic equation is just a (somewhat very!) special case of the equation for a quadric surface, namely $b = c = c = e = f = h = k = 0$. The loci of the

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2D quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

are called the *conics* because these curves are all obtained by cutting a (double) cone in 3D by a plane (you learned that in high school, right?)

One approach to solving quadratic equations is very geometrical (as will be apparent much later.) It consists of turning the quadratic equation into a *homogeneous* equation. We do this by first setting a new variable $w = 1$ and then mutliplying through by 1 a number of times.

Let's see how this works in 1D. We write $ax^2 + bxw + cw^2 = 0$. Note that is is a special case for the 2D quadratic equation ($d = e = f = 0$), provided we release $w = 1$ and let it be 2nd variable in 2D (x, w) space. (Our letters may vary from paragraph to paragraph, but not their roles. Letters at the end of the alphabet are spacial coordinates, and the parameters are at the beginning of the alphabet.)

We now solve the 1D homogogenous quadratic equation as follows.

If $a = 0$ and $b \neq 0$ (if it were, we'd have nothing to look at, right?) then the LHS factors into $w(bx+cw) = 0$, which is true if $w = 0$ or if $(bx + cw) = 0$, both of which are equations of straight lines in (x, w) space. To get back to 1D, we set $w = 1$ (which is still another line) and look at the intersection the locus (loci) makes with the special line $w = 1$.

If $a \neq 0$ then we can eliminate it by dividing through the equation. Note that the RHS $0/a = 0$ doesn't change, so the locus can't change either when we solve

$$\begin{aligned} x^2 + 2\frac{b}{2a}xw + \frac{c}{a}w^2 &= 0 \\ x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}w^2 &= (\frac{b^2}{4a^2} - \frac{c}{a})w^2 \\ (x + \frac{b}{2a})^2 &= (\frac{\sqrt{b^2 - 4ac}}{2a}w)^2 \\ (x + (\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a})w)(x + (\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a})w) &= 0 \end{aligned}$$

The last line is the equation of two lines crossing at the origin. When we set $w = 1$ again, these two lines cross this horizontal at the two solutions to the quadratic equations.

The foregoing was an example of how to solve an inhomogenous equation in some dimensions, by solving a homogeneous equation in one dimension higher. Working in homogenous coordinates is an essentail aspect of computer graphics as it has been in algebraic geometry for hundreds of years.

Note that by solving the 1D quadratic equation in the foregoing way, also solved the special case of the 2D quadratic equations $x^2 + bxy + cy^2 = 0$, only with different letters. We next consider the "latter half" of the entire quadratic equation, namely the case of *linear equations* in the plane, $dx + ey + f = 0$, except that we rewrite this equation again, as $ax + by + c = 0$. This time we want to know how to characterize each line such an equation defines. In high school you knew lines by their "point-slopes", or by two points the line passes through, or some other geometrical property of the line. Now we want to understand the lines in the plane as the *points* in some space, which we shall call the *moduli space* of lines in the plane. More familiar names for this might be *configuration space*. In the present case, it is not the same as the *parameter space*, which is the 3D a, b, c space. Why? Because the point (a, b, c) is not unique to the line. The equation $tax + tby + tc = 0$ describes the exact same line, provided that $t \neq 0$, of course. (Think: what is the locus when $t = 0$.) The Greeks would have said that it isn't the triple (a, b, c) that determines the line, but their ratio $a : b : c$. So, now we want to know how to imagine the space of all ratios $a : b : c$, except $a = b = c = 0$.

$$\pounds = \{a : b : c | 0 \neq a^2 + b^2 + c^2 = r^2\} \text{ and } a : b : c = \{(ta, tb, tc) | t \neq 0\}$$

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To solve this visualization problem we shall need some topology.