Fractal Dimensions and the Sierpinski Tetrahedron (Proposal Draft)

Wyatt Newberry

October 2015

1 Introduction

A fractal could be considered as a mathematical idea of an entity that technically exhibits (but not exactly) self-similarity on every scale. The mathematical concept of a fractal was discovered by French mathematician Gaston Julia. Julia was especially interested in iterative sets of functions. In 1918, Julia published a mathematical paper, *"Mémoire sur l'itération des fonctions rationnelles"*describing iterations of functions, leading to a very important discovery.[1]

2 The Julia Set

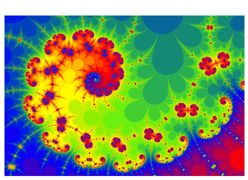
Some of Julia's work focused specifically on iterations of a rational function. Julia chose some rational function, and generated a transformation from quadratic mapping to depict his Julia sets. Mathematically, the rational function is represented as: Let:

$$R(z) = \frac{P(z)}{Q(z)} \text{ where } z \in C$$

and P and Q are polynomials without common divisors. The quadratic map Julia sets are generated with is:

$$z_n + 1 = (z_n)^2 + c$$

for a fixed value c. When quadratic mapped, these iterations generate very interesting patterns. For example:

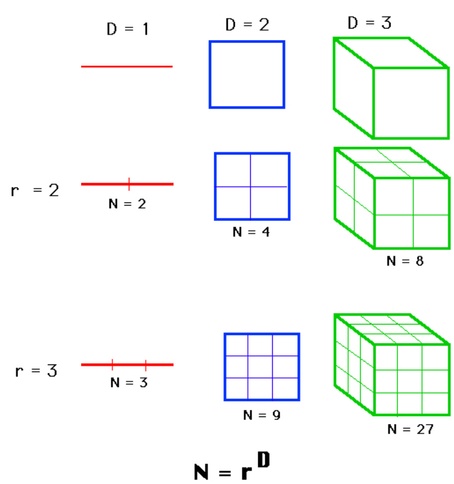


[1]Gaston Julia Wikipedia Page, `http://en.wikipedia.org/wiki/Gaston_Julia`

Taking a closer look at the image, one can soon realize this a lot more than a pretty picture. This image has repeating patterns (or it could be considered self-similar) and another quality that I haven't mentioned yet. Notice how all of points in the system tend to follow the same pattern, and tend to be in similar spots relatively throughout the image. This is no mistake. This depicts another mathematical term related to fractals, an attractor. An attractor is a set of numerical values that a system tends to take the shape of.[2] The attractor is different depending entirely on the system, and the functions used for the iteration in the julia set. All of these factors prove that the image produced by the iteration of a julia set are (almost) all fractals.

3 The Fractal Dimension

Another unique property of fractals is how they lie in dimensions. Fractals do not follow the the traditional pattern of 1, 2, 3, 4, etc. dimensions. Fractals often lie between these standard dimensions. How is this possible? Well, lets start with standard Euclidean dimensions. If the linear side of an object residing in Euclidean Dimension D is reduced by $\frac{1}{r}$ in each spatial direction, its unit of measurement (e.g. length, area, or volume) would increas to $N = r^D$ times the original.[3] This is easily represented in this picture:



[2]Wolfram MathWorld, `mathworld.wolfram.com/Attractor.html`

[3]Vanderbilt, Fractals and the Fractal Dimension `http://www.vanderbilt.edu/AnS/psychology/cogsci/chaos/workshop/Fractals.html`

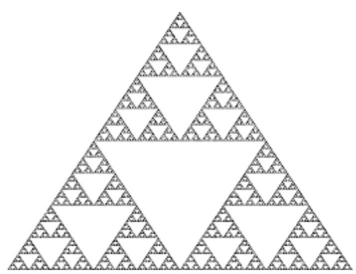
In the case of fractals, their dimensions are denoted by the term "Hausdorff Dimension". To calculate a specific fractal's Hausdorff dimension, a simple formula can be followed:

$N = s^d$ and $d = \frac{ln(N)}{ln(s)}$ where $N =$ (number of self-similar pieces) and $s =$ (magnification factor) of each division of the fractal.[4]

While these fractional dimensions aren't clearly obvious to our eyes, we can still see them indirectly. These Hausdorff dimensions serve as the "invisible" boundary to the Julia set fractals.

4 The Sierpinski Triangle and Tetrahedron

The Sierpinski Triangle is a fractal and attractive fixed set that is overall an equilateral triangle divided into small equilateral triangles as shown here:



This triangle divides into 3 self-similar pieces, and has a magnification factor of 2. Therefore, its Hausdorff dimension is $\frac{ln(3)}{ln(2)} \approx 1.58$. This triangle can be generated easily in two main methods. Method one is the chaos method, which generates the triangle similar to how a Julia set is generated. The corners of the triangles could be considered as p_1, p_2, p_3 and a random point could be named v_1. Then, set $v_(n+1) = \frac{1}{2}(v_n + p_{rn})$ where r_n is a random number 1, 2, or 3. Then this is iterates for values v_1 to v_∞. If the original point v_1 in on the triangle, then all points v_n will lie on the triangle. If v_1 that is within the perimeter of the triangle is not on the Sierpinski triangle, none of the points will be on the triangle. But, these points will converge on the triangle due to the attractor.[5]
Another method of generating this triangle is via recursion. Recursion is at the most basic definition a repeated application of a procedure. This method is often used in computer science to generated this type of fractal. A set of instructions can be provided as code that divides the triangle, determines the new vertices of the interior triangles, and recursively call these instructions to implement the fractal. Due to the fact it is already clear how to implement

[4]Boston University Mathematics, `http://math.bu.edu/DYSYS/chaos-game/node6.html`
[5]Sierpinski Triangle, `https://en.wikipedia.org/wiki/Sierpinski_triangle#Chaos_game`

the Sierpinski Triangle, my project idea is to bring this concept to a higher dimension. I want to implement the Sierpinski Tetrahedron, the 3 space version of the Sierpinski Triangle. I will create this fractal via the recursive method using a combination of Javascript and HTML along with a Javascript library (THREE.JS) used for geometry.