Fractals and the Sierpinski Tetrahedron

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1 Introduction

The definition of a fractal is often the topic of debate among mathematicians. For the sake of this project, I'll be considering fractals as a set with a dimension that is a fraction. The mathematical concept of a fractal was discovered by French mathematician Gaston Julia. Julia was interested in the set of points defined by iteration of functions. In 1918, Julia published a mathematical paper, *Mémoire sur l'itération des fonctions rationnelles*[1] describing iterations of functions, leading to a very important discovery.

2 The Julia Set

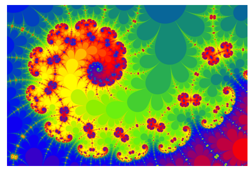
Some of Julia's work focused specifically on iterations of a rational function. A rational function is function that is defined by a rational fraction, such that the denominator and the numerator are polynomials. Julia chose some rational function, and generated a transformation from quadratic mapping to depict his Julia sets. Mathematically, the rational function is represented as: Let:

$$R(z) = \frac{P(z)}{Q(z)} \text{ where } z \in C$$

and P and Q are polynomials without common divisors. The quadratic map Julia sets are generated with is:

$$z_{n+1} = z_n^2 + c$$

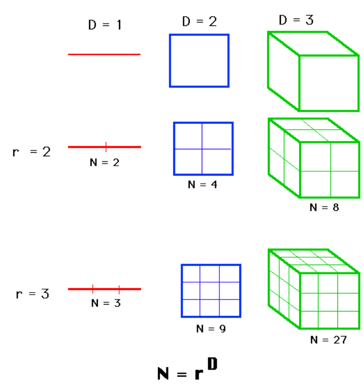
for a fixed value c. A quadratic map is the points that represent the set of the iteration of a quadratic function. When quadratic mapped, these iterations are used to color pixels to produce interesting patterns. For example:



Taking a closer look at the image, one can soon realize that this image a lot more than a pretty picture. This image has repeating patterns (or it could be considered self-similar) and another quality that I haven't mentioned yet. Notice how all of points in the system tend to follow the same pattern, and tend to be in similar spots relatively throughout the image. This is no mistake. This depicts another mathematical term related to fractals, an attractor. An attractor is a set of numerical values that a system tends to take the shape of.[2] The attractor is different depending entirely on the system, and the functions used for the iteration in the julia set. All of these factors show that the image produced by the iteration of the rational function that converges to a Julia set are fractals.

3 The Fractal Dimension

Another special property of fractals I mentioned earlier is their fractional dimension. Fractals do not follow the the traditional pattern of 1, 2, 3, 4, etc. dimensions. Fractals often lie between these standard dimensions. How is this possible? Well, lets start with standard Euclidean dimensions. If the linear side of an object residing in Euclidean Dimension D is reduced by $\frac{1}{r}$ in each spatial direction, its unit of measurement (e.g. length, area, or volume) would increase to $N = r^D$ times the original.[3] This is easily represented in this picture:



In the case of fractals, their dimensions are denoted by the term 'Hausdorff Dimension'. To calculate a specific fractal's Hausdorff dimension, a simple formula can be followed:

$N = s^d$ and $d = \frac{ln(N)}{ln(s)}$ where:

$N =$ (number of self-similar pieces) and

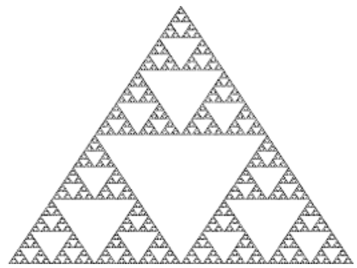
$s =$ (magnification factor) of each division of the fractal.[4]

While these fractional dimensions aren't clearly obvious to our eyes, we can still see them indirectly. The Julia set of a fractal serves as the boundary of what points remain in the set in our view verses what points escape to infinity (which are colored in based on how quickly they escape to infinity).

4 The Sierpinski Triangle and Tetrahedron

The *Sierpinski Triangle* is a fractal and attractive fixed set that is overall an equilateral triangle. To understand the triangle, one must first understand its origin. This fractal is considered a cantor fractal, due to work done by Georg Cantor. Cantor was the man behind the set named after him, the *Cantor Set.* Cantor started with a line, and then divided the line into thirds, and removed the middle segment of the line [5], like so:

What's fascinating about this set is that this process could be done infinitely. This set is actually an example of a fractal with a dimension of $\frac{ln(2)}{ln(3)} \approx 0.631$. So, with this in mind, think about the triangle. It's generated in a very similar method. It begins as an equilateral triangle, and then is split into 4 equal equilateral triangles within that triangle, and the middle triangle is removed. That can be seen here:



Due to the fact this triangle divides into 3 self-similar pieces, and has a magnification factor of 2 its Hausdorff dimension is $\frac{ln(3)}{ln(2)} \approx 1.58$. This triangle can be generated easily in two main methods. Method one is the chaos game (more properly known as an Iterative Function System) method, which generates the triangle similar to how a Julia set is generated. The corners of the triangles could be considered as p_1, p_2, p_3 and a random point could be named v_1. Then, set $v_{n+1} = \frac{1}{2}(v_n + p_{r_n})$ where r_n is a random number 1, 2, or 3. Then this is iterates for values v_1 to v_∞. If the original point v_1 in on the triangle, then all points v_n will lie on the triangle. If v_1 that is within the perimeter of the triangle is not on the Sierpinski triangle, none of the points will be on the triangle. But, these points will converge on the attractor, which is the Sierpinski Triangle.[6]

Another method of generating this triangle is via recursion, which is based upon the cantor method from above. Recursion is at the most basic definition a repeated application of a procedure. This method is used in computer programming to generated this type of fractal. A set of instructions can be provided as code that divides the triangle, determines the new vertices of the interior triangles, and recursively call these instructions to implement the fractal.

5 The Project

Due to the fact it is already clear how to implement the Sierpinski Triangle, my project idea is to bring this concept to a higher dimension. I want to implement the Sierpinski Tetrahedron, the 3 space version of the Sierpinski Triangle. I will create this fractal via the recursive method using a combination of Javascript and HTML along with a Javascript library (THREE.JS) used for geometry.

6 Timetable and Goals

October 19 Implement 2D Sierpinski Triangle with Javascript/HTML Canvas.

October 23 Submit first Draft for Proposal.

October 29 Create example item with THREE.JS library from Graham's tutorials.

November 1 Create Chaos game version of Sierpinski and Chaos game with 4 points.

November 4 Finalize Proposal for seminar and project.

November 6 Finish PowerPoint for Seminar and prepare speaking for seminar.

November 11 Try to have a representation of the Sierpinski Tetrahedron in THREE.JS

November 13 Complete rotation for Sierpinski Tetrahedron Model.

November 16 Finish final documentumentation.

November 18 Verify project is fully functional and ready for final presentation.

November 20 Wrap up project and prepare for final presentation speech/ PowerPoint.

References

[1] Wikipedia, *Gaston Julia* (`http://en.wikipedia.org/wiki/Gaston_Julia`)

[2] Wolfram MathWorld, *Attractors* (`mathworld.wolfram.com/Attractor.html`)

[3] Vanderbilt University, *Fractals and the Fractal Dimension* (`vanderbilt.edu/AnS/psychology/cogsci/chaos/workshop/Fractals.html`)

[4] Boston University Mathematics, *Chaos Game* (`math.bu.edu/DYSYS/chaos-game/node6.html`)

[5] Wikipedia, *Georg Cantor* (en.wikipedia.org/wiki/Cantor_set)

[6] Wikipedia, *Sierpinski Triangle* (en.wikipedia.org/wiki/Sierpinski_triangle#Chaos_game)