A Brief Introduction to Geometric Algebra

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1 Introduction

Vectors as a concept are easy to understand: a direction with a length. They can be represented by arrows, added by aligning them head-to-tail, and scaled by changing their length. Each operation has a nice geometric interpretation.

Multiplying a vector with another vector, however, is less intuitive. Traditionally there have been two different ways to do this. The "dot" product is a number, a scaled projection of one vector onto another, representing how parallel they are. This is a bit unintuitive—why should the product of two vectors be a number?—but it makes sense in physics to measure how closely two vectors align.

The other way to multiply two vectors is the "cross". The product here is a vector, orthogonal to both factors, directed by the right-hand rule, and representing how perpendicular they are. Seems simple enough. But on closer inspection, there are some oddities.

For one, vectors aren't supposed to depend on the space they're in. A vector in two dimensions, in three, in four, all should behave mathematically in the same way. But the idea of a unique direction orthogonal to two others is a peculiarity of three-space. In a plane, there is no such direction, and in a four-space there are an infinite number.

Also, vectors are not supposed to depend on a frame of reference. If I decide that the entire universe is rotated 90 degrees, so x is up and x is east, then all the math should work out the same. But if I reflect everything in a mirror (perhaps by swapping x and $-x$), the cross product suddenly changes direction. This is expected from the right hand rule, but it's unintuitive. We don't expect a vertical mirror to flip something from up to down.

2 History

Over a century and a half ago, the mathematician Hermann Grassmann decided to do something about this. He came up with an alternate vector algebra, based on what he called "das äußeres Produkt", the "Exterior Product". The exterior product of two vectors is not a scalar, or a vector: it is an entirely new kind of beast, called a "bivector". Just as a vector is a directed segment of a line, a bivector is a directed segment of a plane, the parallelogram between the two vectors. The exterior product is anticommutative, like the cross product, and it has the same magnitude. But unlike the cross product it is well-defined in any sort of space and behaves sensibly when reflected—no handedness required.

The problem with the exterior product is that it is not invertible. The product represents the perpendicular component of the two vectors, so either one can be extended parallel to the other without changing the product. This limited the applicability of the new algebra, since there was no way to define division uniquely. Other mathematicians generally thought that Grassmann had some interesting ideas but didn't present them well. Annoyed, Grassmann vowed to abandon mathematics and never return. He became a famous linguist instead and made several important discoveries about Ancient Greek and Sanskrit.

On the other side of the English Channel, British mathematicians were experimenting with the complex numbers. William Rowan Hamilton wanted to find a mathematical way to represent space, the way the complex numbers formed a plane, but he couldn't get it working properly. Just like Grassmann, he couldn't come up with a form of multiplication that would be invertible. And without that, his system lacked the elegance of the complex numbers. One night as he was walking home a flash of inspiration came to him—he needed *three* imaginary units, in addition to the one real unit! He carved the formula into the side of the bridge:

$$i^2 = j^2 = k^2 = ijk = -1$$

And thus the Quaternions were born.

The problem with quaternions is that they are difficult to understand, and not at all simple to visualize. The space of quaternions is four dimensional, but the dimensions aren't all independent, and combine in strange ways. Many other physicists rejected the system, preferring Gibbs' vectors and cross products, and it also mostly fell by the wayside for quite a while.

Eventually came the breakthroughs that would solve all of these problems. William Kingdon Clifford, like Grassmann, tried to build a new algebra with vectors as the basic units. But instead of the exterior product, his algebra was based on the "geometric product" (aka the Clifford product). This is defined to be associative and distributive over addition, and for vectors it ends up being the same as the sum of the inner and outer products: the inner product specifies the parallel component, so unlike Grassmann's product, this is invertible.

$$ab = a \wedge b + a \vee b$$

(for vectors a, b, where $\wedge$ is Grassmann's outer product and $\vee$ is the "inner" (dot) product)

But the inner product is a scalar, and the outer product is a bivector. What does it mean to add them? The second breakthrough was defining a new kind of entity, a "multivector". A multivector is a linear combination of a scalar, a vector, a bivector, a trivector, and so on. Since the geometric product distributes over addition it can be extended to multiply *any* two multivectors, creating the "Clifford algebra" or "geometric algebra".

So what can this geometric product do? It turns out it is very useful for representing transformations in space.

3 Transformations

A bit of background before we begin.

The geometric product is notably invertible, for vectors at least, so we can talk about the multiplicative inverse of a vector:

$$a^{-1} = \frac{1}{a} = \frac{a}{a^2} = \frac{a}{a \vee a + a \wedge a} = \frac{a}{|a|^2}$$

The additive inverse $-a$ is the same as in standard vector algebra, a vector of the same length and opposite direction. Both of these operators are "antiautomorphisms": $(a^{-1})^{-1} = a$, and $-(-a) = a$. There is a third one without a direct parallel in real algebra, the "reverse", defined by two equations:

$$(abc \cdots yz)^{\dagger} = zy \cdots cba$$

(for vectors a, b...) and

$$(a + b)^{\dagger} = a^{\dagger} + b^{\dagger}$$

The utility of this will soon become clear.

So what is the most basic type of transformation in space? We've already been talking about parallel and perpendicular components, while defining the inner and outer products. Let's start with the reflection. A reflection across a particular line[1] is an operation that inverts the perpendicular component without affecting the parallel one. In other words:

$$a' = -a_\perp + a_\parallel$$

We also know that the inner product is zero for orthogonal vectors, and the outer product is zero for parallel ones.

Therefore, if we have a vector n,

$$nvn^{-1} = n(v_\perp + v_\parallel)n^{-1} = nv_\perp n^{-1} + nv_\parallel n^{-1}$$

$$\begin{aligned} nv_\perp n^{-1} &= (n \wedge v_\perp + n \vee v_\perp)n^{-1} \\ &= (n \wedge v_\perp + 0)n^{-1} \\ &= -(v_\perp \wedge n + 0)n^{-1} \\ &= -v_\perp nn^{-1} \\ &= -v_\perp \end{aligned}$$

$$\begin{aligned} nv_\parallel n^{-1} &= (n \wedge v_\parallel + n \vee v_\parallel)n^{-1} \\ &= (0 + n \vee v_\parallel)n^{-1} \\ &= (0 + v_\parallel \vee n)n^{-1} \\ &= v_\parallel nn^{-1} \\ &= v_\parallel \end{aligned}$$

$$nvn^{-1} = -v_\perp + v_\parallel$$

This operation reflects the vector v across the line of the vector n. If n is a unit vector, then $n^{-1} = n$, and this becomes the "sandwich product" nvn.

From here we can derive the equations for rotation. A simple rotation is equivalent to a reflection composed with another reflection, as shown below.

[1] or an $n - 1$-space; I'm focusing on reflections in a plane

$$
\begin{aligned}
v' &= ava \\
v'' &= bv'b = bavab
\end{aligned}
$$

Since the quantity ba here represents the rotation, we can call it a "rotor", R. So this can be restated:

$$v' = RvR^\dagger$$

Rotors are multivectors, so they can be combined using the geometric algebra as well. If we want to compose two rotations, just like we composed two reflections before:

$$
\begin{aligned}
v' &= AvA^\dagger \\
v'' &= Bv'B^\dagger = ABvB^\dagger A^\dagger
\end{aligned}
$$

Since the reversion distributes over multiplication also, the product of these two rotations (also a multivector) can be applied just as reflections and rotations were. In other words, all combinations of those two basic transformations can be applied in the same way. (Translations can also be applied like this in theory through homogeneous coordinates, but I haven't had a chance to experiment with this.)

4 Lining it Up

The simplest form of the Geometric Algebra would be in one dimension, but that is relatively boring. All vectors are scalar multiples of each other and thus it isn't much different from the real numbers. In fact, it's *exactly* the same as the real numbers, just dressed up a bit. I mention it only for completeness.

5 Things Get Complex

So far I've tried to avoid direct reference to coordinates or bases. But for the purposes of numerical calculation they can make things far easier.

Consider a plane. We can choose any two ortho-normal vectors in this plane, and call them $\hat{x}$ and $\hat{y}$. Since we don't need a right hand rule it doesn't matter which two vectors we choose, just so long as they are unit vectors orthogonal to each other. Now any other vector in the plane can be written uniquely as a linear combination of $\hat{x}$ and $\hat{y}$.

What about bivectors? A bivector is a segment of a plane with direction and area. Since there's only one plane in this space, every bivector must lie in it; the direction can be represented by a sign, and the area by a scalar. Therefore every bivector in this space is a scalar multiple (positive or negative) of the basis bivector $\widehat{xy} = \hat{x} \wedge \hat{y}$.

Now we have our vector and bivector bases, and our scalar basis is simply 1. So there are four linearly independent bases for a multivector in this space, and an arbitrary multivector can be written as

$$\alpha 1 + \beta\hat{x} + \gamma\hat{y} + \delta\widehat{xy}$$

(This is the method that my Multivector library uses internally: each multivector object holds a list of real numbers, which are interpreted as the coefficients for the various bases.)

Since the geometric product distributes over addition, writing multivectors in this format makes it easy to multiply them. We just need a multiplication table for the various bases:

$$
\begin{aligned}
11 &= 1 \\
1\hat{x} &= \hat{x} \\
1\hat{y} &= \hat{y} \\
1\widehat{xy} &= \widehat{xy} \\
\\
\hat{x}1 &= \hat{x} \\
\hat{x}\hat{x} &= \hat{x} \wedge \hat{x} + \hat{x} \vee \hat{x} = 1 \\
\hat{x}\hat{y} &= \hat{x} \wedge \hat{y} + \hat{x} \vee \hat{y} = \widehat{xy} \\
\hat{x}\widehat{xy} &= \hat{x}\hat{x}\hat{y} = \hat{y} \\
\\
\hat{y}1 &= \hat{y} \\
\hat{y}\hat{x} &= \hat{y} \wedge \hat{x} + \hat{y} \vee \hat{x} = -\widehat{xy} \\
\hat{y}\hat{y} &= \hat{y} \wedge \hat{y} + \hat{y} \vee \hat{y} = 1 \\
\hat{y}\widehat{xy} &= \hat{y}\hat{x}\hat{y} = -\hat{x}\hat{y}\hat{y} = -\hat{x} \\
\\
\widehat{xy}1 &= \widehat{xy} \\
\widehat{xy}\hat{x} &= \hat{x}\hat{y}\hat{x} = -\hat{y}\hat{x}\hat{x} = -\hat{y} \\
\widehat{xy}\hat{y} &= \hat{x}\hat{y}\hat{y} = \hat{x} \\
\widehat{xy}\widehat{xy} &= \hat{x}\hat{y}\hat{x}\hat{y} = -\hat{y}\hat{x}\hat{x}\hat{y} = -1
\end{aligned}
$$

The last of these is the most interesting: the basis bivector in a plane squares to negative one. We can also look at the reversion operator:

$$
\begin{aligned}
1^\dagger &= 1 \\
\hat{x}^\dagger &= \hat{x} \\
\hat{y}^\dagger &= \hat{y} \\
\widehat{xy}^\dagger &= (\hat{x}\hat{y})^\dagger \\
&= \hat{y}\hat{x} \\
&= -\hat{x}\hat{y} \\
&= -\widehat{xy}
\end{aligned}
$$

Since it distributes over addition, the reversion operator (in a plane) inverts bivectors while leaving scalars and vectors untouched.

And now for the coup de grâce. Recall from earlier that a rotor is a geometric product of two vectors. If we write this in terms of the basis…

$$(\alpha\hat{x}+\beta\hat{y})(\gamma\hat{x}+\delta\hat{y}) = \alpha\gamma\hat{x}\hat{x}+\alpha\delta\hat{x}\hat{y}+\beta\gamma\hat{y}\hat{x}+\beta\delta\hat{y}\hat{y} = (\alpha\gamma+\beta\delta)1+(\alpha\delta-\beta\gamma)\widehat{xy}$$

So a rotor is the sum of a scalar and a bivector. The bivector basis is linearly independent with the scalar basis. It squares to -1. And the reversion operator inverts it while preserving the scalar part.

With this in mind, why not adjust our terminology a bit?

$$\begin{aligned} \widehat{xy} &= i \\ \alpha 1 + \delta\widehat{xy} &= a + bi \\ \mathbf{a}^\dagger &= \overline{z} \end{aligned}$$

It turns out that the rotors in a plane, multiplied with the geometric product, are actually isomorphic to the complex numbers.

6 Geometric Algebra...IN SPACE

In three dimensions things are even more interesting. As before, we can pick some ortho-normal vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ for our basis, without worrying about handedness. This time there are three different ways to combine the basis vectors: $\widehat{xy}$, $\widehat{yz}$, and $\widehat{zx}$. And we have a new type of element that didn't exist in two dimensions, the 'trivector' $\widehat{xyz}$.

The multiplication table can be derived as in two dimensions. (First factor on the left, second across the top.)

	1	$\hat{x}$	$\hat{y}$	$\hat{z}$	$\widehat{xy}$	$\widehat{yz}$	$\widehat{zx}$	$\widehat{xyz}$
1	1	$\hat{x}$	$\hat{y}$	$\hat{z}$	$\widehat{xy}$	$\widehat{yz}$	$\widehat{zx}$	$\widehat{xyz}$
$\hat{x}$	$\hat{x}$	1	$\widehat{xy}$	$-\widehat{zx}$	$\hat{y}$	$\widehat{xyz}$	$-\hat{z}$	$\widehat{yz}$
$\hat{y}$	$\hat{y}$	$-\widehat{xy}$	1	$\widehat{yz}$	$-\hat{x}$	$\hat{z}$	$\widehat{xyz}$	$\widehat{zx}$
$\hat{z}$	$\hat{z}$	$\widehat{zx}$	$-\widehat{yz}$	1	$\widehat{xyz}$	$-\hat{y}$	$\hat{x}$	$\widehat{xy}$
$\widehat{xy}$	$\widehat{xy}$	$-\hat{y}$	$\hat{x}$	$\widehat{xyz}$	-1	$-\widehat{zx}$	$\widehat{yz}$	$-\hat{z}$
$\widehat{yz}$	$\widehat{yz}$	$\widehat{xyz}$	$-\hat{z}$	$\hat{y}$	$\widehat{zx}$	-1	$-\widehat{xy}$	$-\hat{x}$
$\widehat{zx}$	$\widehat{zx}$	$\hat{z}$	$\widehat{xyz}$	$-\hat{x}$	$-\widehat{yz}$	$\widehat{xy}$	-1	$-\hat{y}$
$\widehat{xyz}$	$\widehat{xyz}$	$\widehat{yz}$	$\widehat{zx}$	$\widehat{xy}$	$-\hat{z}$	$-\hat{x}$	$-\hat{y}$	1

Rotors are pretty much the same as in two dimensions, but multiplying them out gives four terms instead of two:

$$(\alpha\hat{x} + \beta\hat{y} + \gamma\hat{z})(\delta\hat{x} + \epsilon\hat{y} + \zeta\hat{z}) = \zeta 1 + \eta\widehat{xy} + \theta\widehat{yz} + \kappa\widehat{zx}$$

Once again an isomorphism emerges. The basis bivectors again square to -1, but now there are three instead of one. If we rename them as i, j, and k...

$$i^2 = j^2 = k^2 = ijk = -1$$

This is Hamilton's famous equation defining the Quaternions. If we also rename the reversion as the "conjugate", the formula for quaternion rotations also emerges from this. In other words the real numbers, the complex numbers, and the Quaternions are all special cases of the Geometric Algebra (in one, two, and three dimensions).

7 The Hodge Dual

The last column in the multiplication table shows something interesting. Multiplying any basis by $\widehat{xyz}$, the "unit antiscalar", gives its opposite: the basis which contains none of the same basis vectors. So $\hat{x}$ becomes $\widehat{yz}$ and vice-versa. It turns a scalar into a trivector, a vector into a bivector, a bivector into a vector, and a trivector into a scalar.

This operation is referred to as the Hodge dual, or the Geometric Star.

$$\widehat{xyz}\hat{x} = \star\hat{x} = \widehat{yz}$$

Since $\widehat{xyz}$ squares to 1, the Star is its own inverse, like the negative and reverse operators.

The main use of this operator is actually hiding in plain sight. It's effectively a multiplication, so it distributes over addition and affects each term in a multivector. In three dimensions it turns a bivector into a straight vector and vice-versa, in a way that preserves magnitude. Since the three vector bases are orthogonal, $\star\vec{a}$ is orthogonal to $\vec{a}$. And choosing $\star = \widehat{xyz}$ over $\widehat{zyx}$ or some other arrangement imparts a chirality on the space.

In other words, the Star is the operation underlying the cross product.

$$\vec{u} \times \vec{v} = \star(\vec{u} \wedge \vec{v})$$

Alas, it only has this property in three-space. In general the Star turns a k-ary vector into an $(n-k)$-ary vector, so the correspondence between vectors and bivectors only holds when n is 3.

8 ...And Beyond?

The number of bases in different dimensions has so far followed a pattern.

Dimensions	Scalars	Vectors	Bivectors	Trivectors
0	1			
1	1	1		
2	1	2	1	
3	1	3	3	1

In other words, there are $\binom{n}{k}$ ways to create a k-ary basis in n dimensions. So the four-dimensional equivalent to the complex numbers and quaternions would seem to have seven terms (one scalar and six bivectors).

However, this is not the case. There's no "Septimion" algebra[2]. In three dimensions and below, every bivector is "simple": it can be written as the wedge product of two vectors. Or in other words, every bivector lies in a plane. In four dimensions, things change.

To use an analogy, consider two lines. In a one-space they have to be the same line. In a two-space they have to be parallel or intersect. But in a three-space, they may also be skew: neither parallel or intersecting, just totally separate and not interacting at all.

The same thing happens with bivectors. In a two-space, all bivectors are in the same plane. In a three-space they are either coplanar or they can be added edge-to-edge, giving a new bivector in a plane. But in a four-space, what happens if we add $\widehat{xy}$ and $\widehat{zw}$? There's no way to draw these two bivectors so that they share an edge. There's no way to write this as a single product: the best we can do is $\hat{x} \wedge \hat{y} + \hat{z} \wedge \hat{w}$.

We can still represent rotors in the same way. As predicted, there are seven terms.

[2]There are the Octonions, but they aren't associative so they don't count.

$$\alpha 1 + \beta\widehat{xy} + \gamma\widehat{xz} + \delta\widehat{yz} + \epsilon\widehat{xw} + \zeta\widehat{yw} + \eta\widehat{zw}$$

Since it isn't possible to have more than four mutually orthogonal lines in four-space, it isn't possible to have more than two disconnected planes of rotation. In six dimensions or more it is possible to have three, in eight dimensions four, and so on. In general, an n-dimensional rotation can consist of up to $\left\lfloor \frac{n}{2} \right\rfloor$ simple rotations.

9 Physics

For my RTICA I simulate some basic angular dynamics using geometric algebra. The standard formulation involves viewing torques and angular momenta as cross products, so it works only in three dimensions. But by reworking the equations in terms of Geometric Algebra they can be extended to n-dimensional Euclidean space.

There are four main quantities to think about: rotation, angular velocity, torque, and moment of inertia. Rotation is the simplest of these. The sum of a bivector and a scalar is a rotor, which can rotate a point as discussed above. Angular velocity is the derivative of this, which I can numerically integrate (by adding ωdt to θ every timestep, where dt is a very small unit of time).

Torque and moment of inertia are more interesting. Torque is now a bivector, the outer product of the force and moment arm. Traditionally, moment of inertia is defined around a particular axis of rotation. But the axis of rotation only exists in three dimensions. In two dimensions there is no axis perpendicular to a given rotor, and in four dimensions there are infinitely many.

The solution is to project the entire object into the plane of rotation (the plane defined by the torque bivector), then integrate the distance in that plane from each point to the center of rotation. This projection is accomplished easily by the inner product.

$$\tau = F \wedge r_F$$
$$I = \int_m r_m \vee \frac{\tau}{|\tau|} dm$$

Once again here I use an approximate numerical integral: the object is divided up into a finite number of points, the same ones used for rendering, and the mass is evenly divided between them. This gives a compromise between accuracy and run-time speed.

10 Conclusion

This has been a very brief overview of the Geometric Algebra, showing the basic concepts behind it. The algebra itself is fairly simple. But it is also incredibly powerful: the commonly-used real numbers, complex numbers, and Quaternions are all merely special cases. It also generalizes vector mathematics to higher dimensions in a way that traditional vector algebra cannot. Even fields as separate as quantum electrodynamics can be explained more easily with Geometric Algebra[3].

I've created a Python library for working with multivectors in any number of Euclidean dimensions, another to simulate linear and angular dynamics on rigid bodies using it, and two demonstrations using them. The first simulates a set of bars from $(0, 0 \ldots 0)$ to $(1, 0 \ldots 0)$ to $(1, 1 \ldots 0)$ and so on, hinged at the origin. The user can apply random forces at random positions to watch it spin, or combine unit rotations to see how certain combinations look. The second simulates a gyroscope in three dimensions held at one side. The combination of forces causes it to precess slightly, showing how the Geometric Algebra can also be used for simple physical systems in three dimensions.

Hopefully through the use of these programs the concepts of the Geometric Algebra can become more familiar and usable, especially for physics in spaces other than three Euclidean dimensions.

[3]The basic Pauli matrices are isomorphic to the Quaternions, with the Hermitian transjugate isomorphic to the reversion operator.