Liouville's Theorem for Hamiltonian Systems

Samuel Stephens

October 17, 2014

Abstract

Physical systems can be described in many ways, one of the most significant being the Hamiltonian. This formulation gave rise to Liouville's theorem, a theorem about reversability in classical systems. This project will explore the meaning of this theorem through visualizations of phase space and numerically validate it for the cases of a pendulum and a mass on an inclined wedge allowed to slide.

Liouville's theorem states that phase space has a divergence of zero. An equivalent but more insightful interpretation is that any volume of phase space, when evolved, will maintain its volume. In my project, I will numerically validate this statement. To do this, I will begin with a small box of this phase space, let it evolve for a small amount of time, and find the new volume. To find this volume, I use the determinant of the matrix who's columns are the change vectors for the different starting configurations and find its determinant. This assumes that the new volume is a parallelotope, but this will be accurate for small enough changes in time. Visually, I will plot many (on the order of dozens) of different phase space points in small regions and let them evolve with time, to visually show that they maintain their volume.

I will use Python OpenGL. The equations of motion are single order differential equations which I will numerically solve using RK4. The user will choose the initial state or set of states to evolve, and the program will show the initial configurations moving in phase space, represented as a 2D plane. Parallel with this role, the program should also validate that volume is conserved by evolving nearby points in phase space initially positioned as a cube for small amounts of time and recalculating the volume. I would like to be able to show that volume is conserved more accurately by using many initial points so longer time frames can be used, but this may be too slow to calculate.

Timeline:
10/27: Implement RK4 in python for the given examples
11/03: Measure the change in volume for a small initial box
11/17: Implement the visualization of phase space
12/01: Allow the user to set parameters
12/08: Create the webpage

Bibliography:

1. http://theoreticalminimum.com/courses/classical-mechanics/2011/fall

2. http://en.wikipedia.org/wiki/Liouville's_theorem_(Hamiltonian)

3. http://www.nyu.edu/classes/tuckerman/stat.mech/lectures/lecture_2/node2.html