Liouville's Theorem for Hamiltonian Systems

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Math 198
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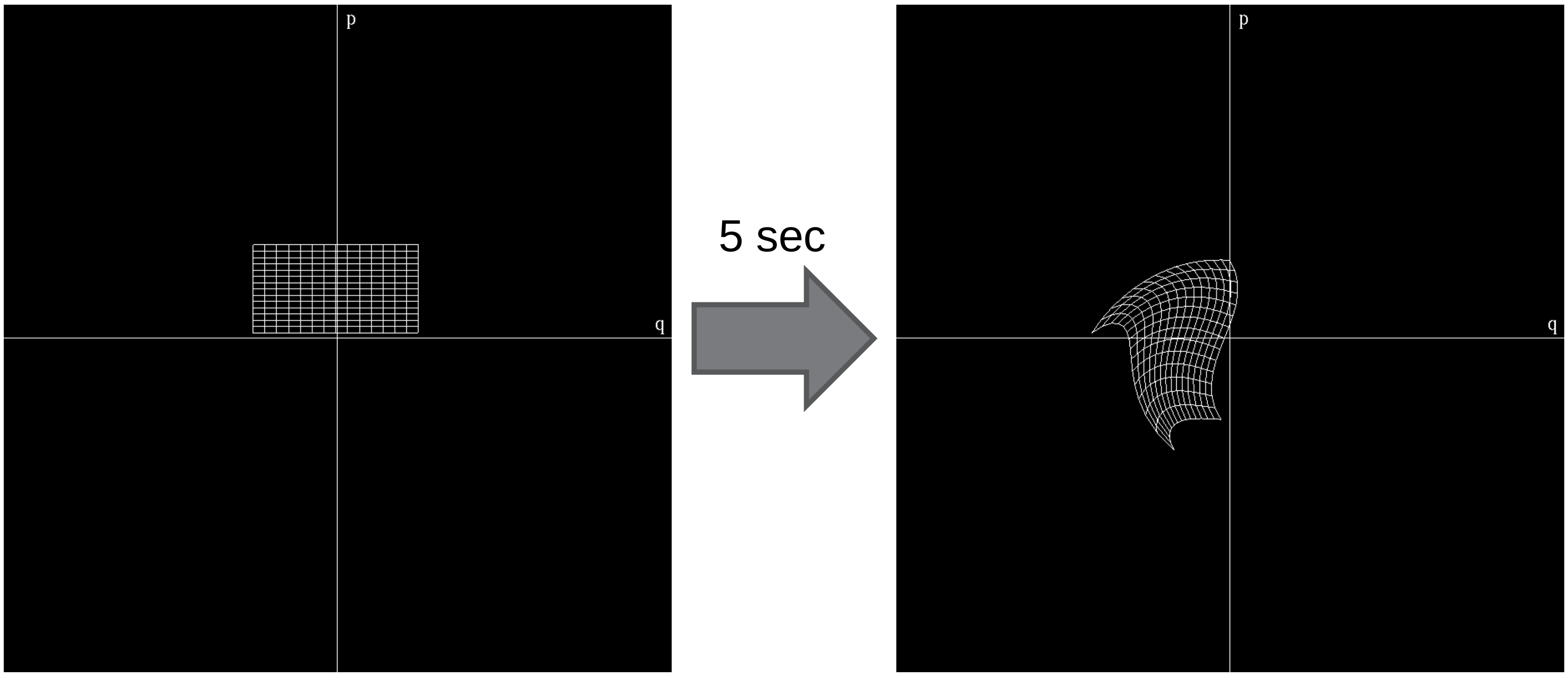
What is Liouville's Theorem?

- The *phase* space of a system is the set of all possible configurations, or states
  - For physical systems, phase space is labeled by a set of positions and momentums as coordinates
  - For example, a fixed-length pendulum requires an angle (q) and an angular momentum (p)
  - As a state evolves, it moves through phase space
- Liouville's theorem states volumes are conserved in phase space
- The goal is to numerically validate Liouville's theorem and show it visually

What does the program do?

- Draws two dimensional phase space
  - Many states are plotted
  - Updates each state individually for small time increments
  - Visually, an initial set of states will appear to maintain its area (according to Liouville’s Theorem)
- Numerically calculates the volume of a region of phase space at each instance of time
- User interaction
  - Select the system
  - Select regions of phase space to fill with states
  - Basic controls, like restart, pause, or change speed

Before and After for the Pendulum



Area = 2.24

Area = 2.15

The Systems

- Pendulum
  - p is angular momentum
  - q is angle from vertical
- Gravity Cart
  - Cart on a fixed, horizontal track above a gravitational mass
  - p is horizontal momentum
  - q is horizontal displacement
- Double Pendulum
  - p_1, p_2 are angular momentum for each pendulum
  - q_1 is the first pendulum angle from horizontal
  - q_2 is the second pendulum angle from the first

How the Program Works

- Each system has a Hamiltonian function, depending on p and q
- Equations of motion are calculated numerically using Hamilton's equations:

$$\frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial q_i}$$

$$\frac{\partial q_i}{\partial t} = +\frac{\partial H}{\partial p_i}$$

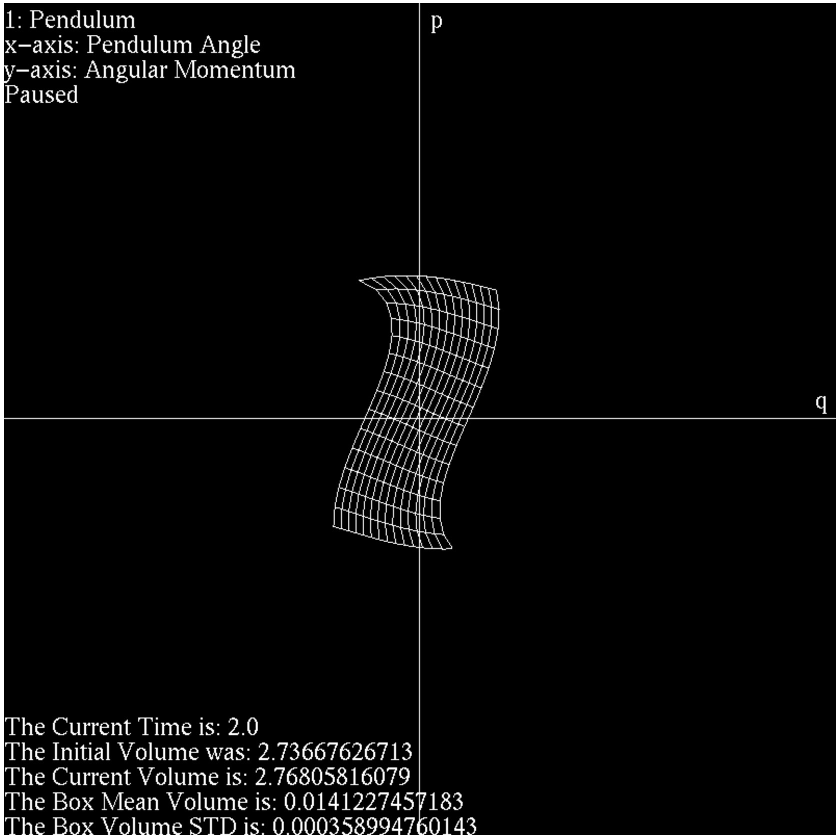
- These derivatives could be calculated exactly for the current systems, but numerically calculating them allows any Hamiltonian to be used

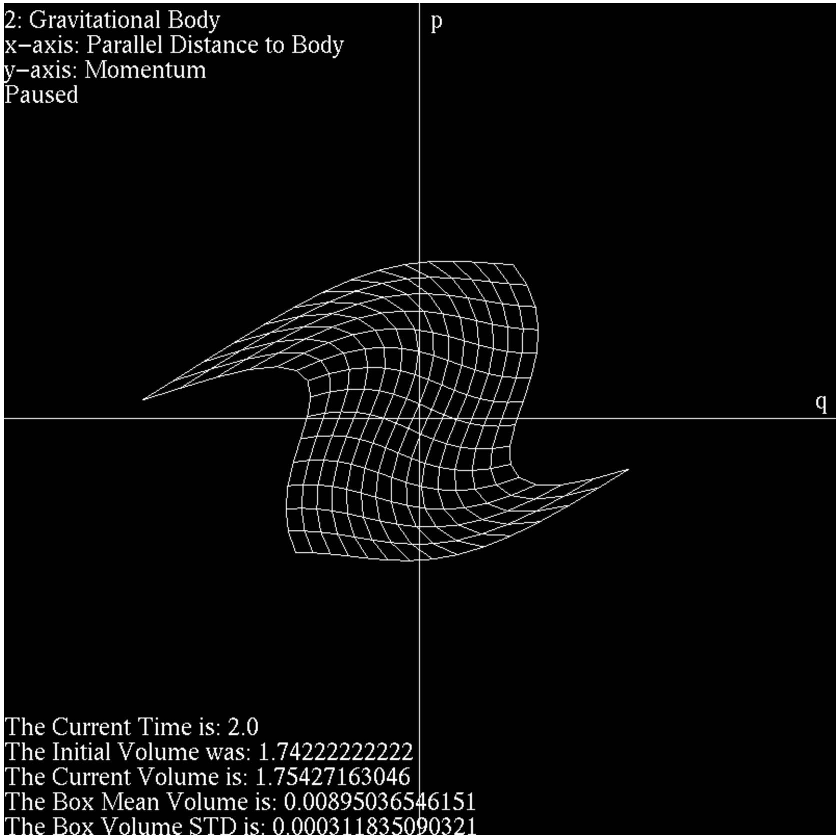
How the Program works (cont.)

- RK4
  - Hamilton’s equations give differential equations for the changes in p and q
  - RK4 can be used to update these values for small time steps
- Volume Calculation
  - The program begins with a grid of states and maintains the grid lines as the system evolves
  - The volume of the total region is calculated by summing the volumes of each smaller region
  - Each smaller region is assumed to be a parallelogram for calculation purposes

Accuracy

- Double Pendulum
  - Time: .5 seconds
  - Initial Volume: .01500625
  - Final Volume: .01533236
  - 2.17% error
- Used very low simulation speed
- Representative samples of phase space used
  - Very small regions will have very little error
  - Regions can’t include diverging points (e.g. regions where some states of the pendulum flip, other states do not)





Pendulum: 1.15% error

Gravity Cart: 0.69% error

Accuracy could always be improved by creating more sub-regions and smaller time increments

Problems

- Fails at diverging points
  - For the pendulum, if one state has enough energy to complete a revolution, while a neighbor state does not, the volume calculation fails, and the grid lines clutter the screen
- Only good at showing oscillating systems
  - Other systems, like a falling ball, will have all states leaving the screen
- Very slow
  - Because each derivative and integral is calculated numerically, only around a hundred states can be simulated at once
  - Systems in more than two dimensions lose accuracy rapidly

Meaning/Application

- If a state is known to be in a small volume of phase space, that state is known very precisely
  - At any other point in time, that state can still be known precisely
- Began the idea of information as a fundamental concept in physics, especially in quantum mechanics
  - The uncertainty principle is a lower bound on the size of these volumes, given by: $\Delta q \Delta p \geq \frac{\hbar}{2}$
- The program's simulation of phase space can reveal other properties of the system

Future Improvements

- Better visualization of the phase space
  - Show color
  - Able to zoom in/zoom out
  - Draw tails
  - Project the double pendulum phase space
- More dimensions
  - Currently only support up to four
- Better numerical methods

Questions?