Liouville's Theorem for Hamiltonian Systems Documentation

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Abstract

Physical systems can be described in many ways, one of the most significant is by their *Hamiltonian function*, an equation for the energy of a system. This formulation gave rise to Liouville's theorem, a theorem about reversibility in classical systems. This project will explore the meaning of this theorem through visualizations of *phase space*, the set of possible states for a system, and numerically validate it for the cases of a pendulum, double pendulum, and a gravitational mass on a fixed track.

1 Introduction

Any physical system has a multitude of different possible states. For example, a pendulum can be angled downward and swinging fast, or stationary but vertical, or more. The phase space for a system is the set of all possible states of that system, labeled by a set of coordinates. For example, the state of a pendulum can be labeled by the angle it makes with the vertical as well as its angular momentum. The method of choosing these coordinates is explained in more detail in the math section. As a state evolves, its location in phase space will move. Liouville's theorem states that volume is conserved in phase space. The program shows this by allowing the user to select a region of phase space and letting it evolve with time, which will (roughly) maintain its volume. The program numerically validates this by calculating the volume of the region at each instance of time.

2 Running the program

To execute the program, run liouvillestheorem.py using python. OpenGL is required.

3 Interacting with the program

3.1 Interpreting output

In the middle of the screen is the plot for many states in phase space, labeled by the coordinates specified in the top left. Note that nothing is drawn for the double pendulum because the phase space is four dimensional. Each point, or corner grid line, represents a state in the phase space, which evolves according to Hamilton's equations. By Liouville's Theorem, the area of any section of phase space should maintain its volume with time.

In the top left, the current system is labeled as well as the axes. In the bottom left are the statistics on the evolution. The current time is the time of evolution, in seconds. The initial volume statistic is the area of phase space when the time was zero, and the current volume is the volume occupied by the phase space at that moment in the simulation. The box mean volume statistic is the average volume of each of the smaller sub-boxes which make up the entire volume, and the box volume STD is the standard deviation of these box volumes. Initially each box is the same size, so the standard deviation is zero. If the simulation were perfect, by Liouville's theorem the box volume STD should always be zero and the current volume and initial volume should always be the same.

3.2 User input

Both the keyboard and mouse are required. The help menu can be open/closed by pressing the h key. More detail on key presses:
h : Pauses simulation, disables drawing, and opens the help menu.
q : Quits the program.
p : Pauses the simulation (automatically occurs when the help menu is opened).
g : Toggle drawing grid lines on and off.

s : Toggle show the text in the top left and bottom left about the simulation.
r : Reset the simulation to time 0.
a : Toggle drawing the axes on the graph.
1,2,3 : Select the system. 1 is for the single pendulum, 2 is for an object on a fixed track above a gravitational body, and 3 is for the double pendulum.
+, - : Increase or decrease the simulation speed. Higher speeds will result in less accuracy.

To use the mouse, select an area of the phase space by by clicking and dragging. The program will fill that area of phase space with sample states in a grid like pattern, which will then evolve with time. This does not work for the double pendulum, as no phase space is shown.

4 The math

4.1 Coordinates

Given a system, a set of coordinates describe every possible state within that system. For example, the state of a fixed-length pendulum can be described by the current angle of the arm and the speed at which the arm is rotating. A point in free space can be described by its x, y, and z coordinates with the time derivatives of each of these coordinates, for a total of 6 coordinates. Notice that for each spatial coordinate, labeled by $q_i(t)$, there is a corresponding time derivative, labeled by $\dot{q}_i(t)$.

4.2 The Lagrangian

The Lagrangian is a way of describing a physical system. Between any time two times, a system will make stationary the action, denoted S, given by:

$$S = \int_{t_1}^{t_2} L(\dot{q}(t), p(t))dt$$

Stationary means that very small variations in the path of the system do not change the value of this integral (similar to minimize or maximizing the value of a function). This is very often called the principle of least action, although this is a slight misnomer, as the action need not be minimized. Here, L denotes the Lagrangian, a function of the current state of the system. In general, $L = T - V$, where L is kinetic and V is potential energy.

4.3 Conjugate momentum

From the Lagrangian comes the idea of conjugate momentum, labeled p_i and defined by:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

For a free particle in a potential field, the classic momentum equation $p = mv$ results:

$$p_i = m\dot{q}_i$$

For a pendulum, this definition give angular momentum:

$$p_i = ml^2\dot{q}_i$$

This momentum is not necessarily conserved. Similarly to how the set of q_i and $\dot{q}_i$ can label a state of a system, the set of q_i and p_i can also label a state. The set of all q_i and p_i for a system is known as the *phase space* for the system.

4.4 Hamiltonian and movement in phase space

The Hamiltonian, H, is defined as:

$$H = \sum_i \dot{q}_i p_i - L$$

This represents the total energy of the system. The Hamiltonian gives equations of motion in phase space:

$$\frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial q_i}$$

$$\frac{\partial q_i}{\partial t} = +\frac{\partial H}{\partial p_i}$$

4.5 Liouville's theorem

Liouville's Theorem states:

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i = 0$$

Less formally, this means volumes are conserved in phase space, and this is what my program will show. If at a given point in time, a state is known very precisely, then the state can always be known precisely, and vice versa.

4.6 Runga Kutta

There is no closed form solution for the exact values of $p(t)$ and $q(t)$ for many Hamiltonians. Instead, the program uses Runge-Kutta to predict the new values for p and q after a small amount of time Δt. Label the state, W of the system with:

$$W(t) = \begin{pmatrix} p_1(t) \\ q_1(t) \\ p_2(t) \\ q_2(t) \end{pmatrix}$$

From Hamilton's equations:

$$\dot{W}(t) = \begin{pmatrix} -\frac{\partial H(t)}{\partial q_1} \\ +\frac{\partial H(t)}{\partial p_1} \\ -\frac{\partial H(t)}{\partial q_2} \\ +\frac{\partial H(t)}{\partial p_2} \end{pmatrix} = f(W(t))$$

To compute $W(t + \Delta t)$:

$$\begin{aligned} k_1 &= f(W(t)) \\ k_2 &= f(W(t) + \frac{1}{2}k_1 \Delta t) \\ k_3 &= f(W(t) + \frac{1}{2}k_2 \Delta t) \\ k_4 &= f(W(t) + k_3 \Delta t) \\ W(t + \Delta t) &\approx W(t) + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}$$

4.7 Volume change

The program calculates the volume of the phase space at each instance of time by evolving a set of parallelotopes and calculating their volume. The volume of a paralleltope with edges a, b, c, d is given by the determinant of the matrix:

$$V = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

5 Further improvements

There are many natural extensions to this project. I tried to find a projection of the 4-dimensional phase space of the double pendulum that would still be meaningful to the project by preserving volumes, but none worked. One improvement is adding a way of showing the double pendulum phase space. I implemented Hamilton's equations at the level of the Hamiltonian, so other systems could be easily added rather than the ones provided. I had originally intended that this project show chaotic systems, so this program could be adapted to show the chaotic nature of the double pendulum with the correct initial conditions.

6 Bibliography

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