Meta-Complex Numbers

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Quadratic Equations

Recall that the roots of a quadratic equation

$$ax^2 + bx + c = 0$$

are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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This is a problem! There aren't any real numbers whose square is negative!

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We make a new number that is defined precisely to solve this equation!
Define the "imaginary number" i such that

$$i^2 = -1$$

Then i and $-i$ are solutions to the equation

$$x^2 + 1 = 0$$

Defining Complex Numbers

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One important definition is that of the conjugate of z, denoted here by z^*. We define the conjugate of z as

$$z^* = a - bi$$

In essence, we're just changing the sign of the imaginary part.

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In essence, we're just changing the sign of the imaginary part.
Now that we've defined these new numbers, what can we do with them?

Addition and Subtraction

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Multiplication

We can also easily define multiplication of complex numbers using the idea of the distributive property. The distributive property simply says

$$a(b + c) = ab + ac$$

Keeping this idea in mind, if we let $z = a + bi$ and $w = c + di$, then we define zw as

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Keeping this idea in mind, if we let $z = a + bi$ and $w = c + di$, then we define zw as

$$\begin{aligned} zw &= (a + bi)(c + di) \\ &= a(c + di) + bi(c + di) \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Division

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$$\frac{z}{w} = \frac{z}{w}\frac{w^*}{w^*} = \frac{zw^*}{c^2 + d^2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

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Note that division is undefined when $c^2 + d^2 = 0$, i.e. when $w = 0$.

Exponentiation

To deal with complex exponents, we need to know a few Taylor series first. Recall that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Exponentiation

If we expand the Taylor series for e^{ix}, we get

$$\begin{aligned} e^{ix} &= 1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \cdots \\ &= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots) + i(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots) \\ &= \cos x + i \sin x \end{aligned}$$

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This equation, known as Euler's Formula, is essential for working with complex exponents. From this formula, we get the equation

$$e^{a+bi} = e^{a}(\cos b + i \sin b)$$

Exponentiation

If we want to take exponents with a complex number as the base, we write

$$(a+bi)^{c+di} = (e^{\log(a+bi)})^{c+di}$$

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If we can write a complex number in this form, the logarithm becomes very easy to find.
We use the *polar form* of a complex number to do this.
Write $a + bi$ as

$$a + bi = r(\cos\theta + i\sin\theta)$$

where $r = a^2 + b^2$ and $\theta = \arctan\frac{b}{a}$. (Note that we have to make sure that θ is in the correct quadrant)
From here, we can find that

$$\log(a + bi) = \log(r(\cos\theta + i\sin\theta)) = \log r + \theta i$$

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Suppose that the polar form of $a + bi$ is given by $re^{i\theta}$. Then we can write this as

$$\begin{aligned}(a + bi)^{c+di} &= e^{(c+di)\left(\log\left(re^{i\theta}\right)\right)} \\ &= e^{(c\log r - d\theta)+i(d\log r + c\theta)} \\ &= r^c e^{-d\theta}\left[\cos\left(d\log r + c\theta\right) + i\sin\left(d\log r + c\theta\right)\right]\end{aligned}$$

Importance of Complex Numbers

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- Complex numbers are used in solving differential equations of the form

$$y'' + by' + cy = 0$$

- Electrical engineers use complex numbers to describe circuits
- And many more!

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Defining the Dual Numbers

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where $a, b \in \mathbb{R}$. We call a the *real part* and b the *dual part*.
As with complex numbers, we define the conjugate z^* of the dual number $z = a + b\varepsilon$ as

$$z^* = a - b\varepsilon$$

Addition and Subtraction

Let $z = a + b\varepsilon$ and $w = c + d\varepsilon$. Define addition as

$$z + w = (a + c) + (b + d)\varepsilon$$

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and define division as

$$\frac{z}{w} = \frac{ac + (bc - ad)\varepsilon}{c^2}$$

Note that division is not defined when $c = 0$.

Exponentiation & Logarithms

Using Taylor series in the same way we developed complex exponentiation, it can be shown that

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From this, it is easy to verify that

$$\log(a + b\varepsilon) = \log a + \frac{b}{a}\varepsilon$$

Note that the logarithm is undefined when $a = 0$.

Exponentiation & Logarithms

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Combining dual exponentiation with the dual logarithm allows us to take exponents of arbitrary bases, similar to what we did with complex numbers. The final result for exponentiation is

$$(a + b\varepsilon)^{c+d\varepsilon} = a^c + a^c \left(d \log a + \frac{bc}{a} \right) \varepsilon$$

Note that the base must have a non-zero real part.

Uses of Dual Numbers

We can use dual numbers to automatically differentiate functions.

Uses of Dual Numbers

Consider some function $f(x)$ and its Taylor series expansion centered at 0

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

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If we evaluate $f(x+\varepsilon)$, we get

$$\begin{aligned} f(x+\varepsilon) &= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} (x+\varepsilon)^n \\ &= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} (x^n + n x^{n-1} \varepsilon) \\ &= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n + \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} n x^{n-1} \\ &= f(x) + \varepsilon f'(x) \end{aligned}$$

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where $a, b \in \mathbb{R}$. We call a the *real part* and b the *split part*.
As with complex numbers, we define the conjugate z^* of the split-complex number $z = a + bj$ as

$$z^* = a - bj$$

Addition and Subtraction

Let $z = a + bj$ and $w = c + dj$. Define addition as

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From this, we can set up a system of equations to relate $x + yj$ to e^{a+bj}, where $a + bj = \log{(x + yj)}$. This system is

$$x = \frac{e^{a+b} + e^{a-b}}{2}$$
$$y = \frac{e^{a+b} - e^{a-b}}{2}$$

Exponentiation & Logarithms

If we solve this system of equations, we get

$$a = \frac{1}{2}\log\left((x+y)(x-y)\right)$$
$$b = \frac{1}{2}\log\left(\frac{x+y}{x-y}\right)$$

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This lets us easily define the logarithm!

$$\log(a+bj) = \frac{1}{2}\log\left((a+b)(a-b)\right) + \frac{1}{2}j\log\left(\frac{a+b}{a-b}\right)$$

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Note that the split complex logarithm is not closed. For example, if $(a+b)(a-b) < 0$, we will get a complex number in our answer. To avoid this, we will only define the logarithm of $a+bj$ when $a > \max(b, -b)$.

Exponentiation & Logarithms

Combining the logarithm with exponentiation base e allows us to take exponents with an arbitrary base. After lots of nasty math, we can define exponentiation as

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$$(a+bj)^{c+dj} = \left(\frac{a+b}{a-b}\right)^{\frac{d}{2}} \left((a+b)(a-b)\right)^{\frac{c}{2}} \left(\cosh k + j \sinh k\right)$$

$$k = \frac{c}{2}\log\left(\frac{a+b}{a-b}\right) + \frac{d}{2}\log\left(a^2 - b^2\right)$$

Uses of Split-Complex Numbers

Split-Complex numbers have uses in physics:

[1] en.wikipedia.org/wiki/Split-complex_number

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Split-Complex numbers have uses in physics:
"Split-complex mulitplication has commonly been seen as a Lorentz boost of a spacetime plane." [1]

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How can we visualize four-dimensional results? There are several options:

- Map outputs to different colors in RGB space

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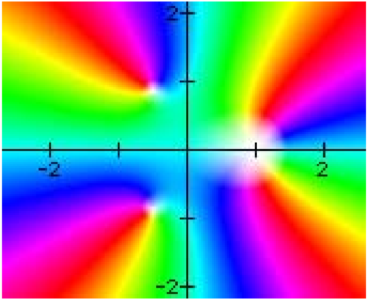
How can we visualize four-dimensional results? There are several options:

- Map outputs to different colors in RGB space
- Plot the output vectors as a vector field in the plane
- Project the four-dimensional surface into 3-space and allow four-dimensional rotations to be made

Color Mappings



A graph of the complex function $(z^3 - 1)(z - 1)$.[2]

[2] Image from www.pacifict.com/ComplexFunctions.html

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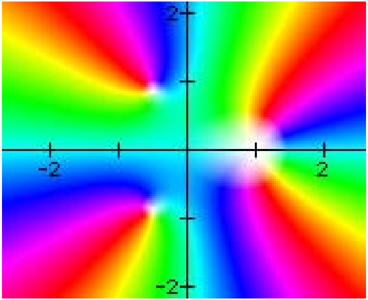
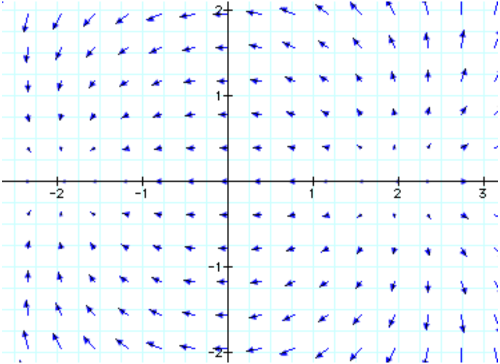


A graph of the complex function $(z^3 - 1)(z - 1)$.[2]

Problem: colors cannot be consistently reproduced on different screens!

[2] Image from www.pacifict.com/ComplexFunctions.html

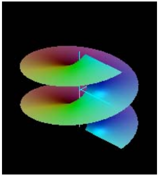
Vector Fields

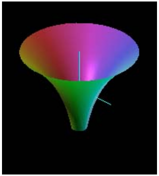


A graph of the complex function $(z+2)(z-2)$.[3]

[3] Image from `www.pacifict.com/ComplexFunctions.html`

Projections into 3-Space





Two projections of the four-dimensional complex exponential into 3-space.[5]

[4] Image from www.pacifict.com/ComplexFunctions.html
[5] Image from www.pacifict.com/ComplexFunctions.html