STICKY SPHERES: ELASTIC AND INELASTIC COLLISIONS IN PYOPENGL

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Abstract. The purpose of this project is to visualize the differences and effects of elastic and inelastic collisions between spheres in 3-space. Sticky Spheres is written in Python and makes use of the PyOpenGL module, and allows variable number of spheres to collide elastically and inelastically. The mathematics behind the code incorporates the equations for conservation of momentum, *perfectly elastic collisions*, and *perfectly inelastic collisions* for two objects of equal mass.

1. Theoretical Collisions

The two types of collisions available in Sticky Spheres are *perfectly elastic collisions* and *perfectly inelastic collisions*. Suppose the spheres have equal mass m, colliding with positions $p_1 = (p_{1x}, p_{1y}, p_{1z})$ and $p_2 = (p_{2x}, p_{2y}, p_{2z})$, initial velocities $v_1 = (v_{1x}, v_{1y}, v_{1z})$ and $v_2 = (v_{2x}, v_{2y}, v_{2z})$, and final velocities $v_1' = (v_{1x}', v_{1y}', v_{1z}')$ and $v_2' = (v_{2x}', v_{2y}', v_{2z}')$. Conservation of momentum means

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

Conservation of kinetic energy means

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 (v_1')^2 + \frac{1}{2} m_2 (v_2')^2$$

1.1. **Perfectly Elastic Collision.**

When two items collide, both the momentum and energy is conserved. Consider the case where $m_1 = m_2$. Then

$$v_1 + v_2 = v_1' + v_2' \text{ and}$$
$$v_1^2 + v_2^2 = (v_1')^2 + (v_2')^2$$

Now, consider the new velocities v_1' and v_2' as the superposition of the new velocities from two seperate collisions: the collision of the first ball moving with velocity v_1 with the stable second ball, and the collision of the second ball moving with velocity v_2 with the stable first ball. In otherwords,

$$v_1'\Big|_{v_1,0} + v_1'\Big|_{0,v_2} = v_1'$$

and

$$v_2'\Big|_{v_1,0} + v_2'\Big|_{0,v_2} = v_2'$$

Consider the first term of the two expressions. That is, consider the first ball has velocity v_1 colliding with a stable second ball. Now, consider a plane made by the location the two balls collide[1]. This plane has a unit normal vector $n = \frac{(p_1 - p_2)}{|(p_1 - p_2)|}$. The velocity of the first ball can be broken into the sum

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[1]The balls are colliding if the distance between the two balls is less than $2 * r$ where r is the radius, that is, $p_1 - p_2 \leq 2r$. The location of collision p_0 is at $p_0 = \frac{p_1 + p_2}{2}$

of two components: the velocity orthogonal (or normal) to the plane, and the velocity parallel to the planel.

$$
\begin{aligned}
v_1^{\perp} &= proj_n\left(v_1\right) = \frac{v_1 \cdot n}{|n|}\frac{n}{|n|} = \left(v_1 \cdot n\right) n \\
v_1^{||} &= v_1 - v_1^{\perp}
\end{aligned}
$$

The final velocities of the first and second ball come exclusively from the first ball. Consider the possibility that the second ball's velocity becomes the parallel velocity component of the first ball and the first ball's velocity becomes the velocity component of the first ball. That is,

$$v_1' = v_1^{||}$$

and

$$v_2' = v_1^{\perp}$$

For these to be correct, they must satisfy the conservation of momentum and conservation of kinetic energy equations

$$
\begin{aligned}
v_1 &= v_1' + v_2' \text{ and} \\
v_1^2 &= (v_1')^2 + (v_2')^2
\end{aligned}
$$

Using algebra:

$$
\begin{aligned}
v_1 &= v_1^{\perp} + v_1^{||} \\
&= v_1' + v_2' && \checkmark \\
\left(v_1\right)^2 &= (v_1^{||} + \left(v_1^{\perp}\right)^2 \\
&= \left(v_1^{||}\right)^2 + 2\left(v_1^{||} \cdot v_1^{\perp}\right) + \left(v_1^{\perp}\right)^2 \\
&= \left(v_1^{||}\right)^2 + 0 + \left(v_1^{\perp}\right)^2 \\
&= \left(v_1'\right)^2 + \left(v_2'\right)^2 && \checkmark
\end{aligned}
$$

Thus,

$$
\begin{aligned}
v_1'\Big|_{v_1,0} &= v_1^{||} \text{ and} \\
v_2'\Big|_{v_1,0} &= v_1^{\perp}
\end{aligned}
$$

Using the same steps, $v_1'\Big|_{0,v_2}$ and $v_2'\Big|_{0,v_2}$ are evaluated to

$$
\begin{aligned}
v_1'\Big|_{0,v_2} &= v_2^{\perp} \text{ and} \\
v_2'\Big|_{0,v_2} &= v_2^{||}
\end{aligned}
$$

Therefore,

$$v_1' = v_1^{\|} + v_2^{\perp}$$
$$v_2' = v_1^{\perp} + v_2^{\|}$$

1.2. **Perfectly Inelastic Collisions.** When two items collide, the momentum is conserved and the energy is not. To be perfectly inelastic, the two items final perpendicular velocity components must be identical. That is,

$$v_1 + v_2 = v_1' + v_2' \text{ and}$$
$$v_1^{\perp\prime} = v_2^{\perp\prime}$$

Analyzing this by components,

$$\begin{aligned} v_1^{\|\prime} &= v_1^{\|} \\ v_2^{\|\prime} &= v_2^{\|} \\ v_1^{\perp\prime} + v_2^{\perp\prime} &= v_1^{\perp} + v_2^{\perp} && \text{(by conservation of momentum)} \\ v_1^{\perp\prime} + v_2^{\perp\prime} &= 2v_1^{\perp\prime} = 2v_2^{\perp\prime} \\ v_1^{\perp\prime} = v_2^{\perp\prime} &= \frac{v_1^{\perp} + v_2^{\perp}}{2} \end{aligned}$$

Thus,

$$v_1' = v_1^{\|} + \frac{v_1^{\perp} + v_2^{\perp}}{2}$$
$$v_2' = v_2^{\|} + \frac{v_1^{\perp} + v_2^{\perp}}{2}$$

2. Verifying Results

Sticky Spheres models the equations for collisions of two spheres of equal mass for both types of collisions. A challenge arose in verifying the equations are correct numerically because of the difficulty of calculating both the number of collisions in a given length of time as well as the relationship between number of collisions, number of spheres, and length of time. However, the lack of conservation of kinetic energy verified that the inelastic collisions did what was predicted.

3. Use of Results

In order to implement elastic and inelastic coillisions in 3-space, the spherical coordinate system was unnatural to program with. Therefore, I derived the equations in terms of vectors without trigonometry. Because the equations are independent of the number of dimensions, these same equations can be used to implement collisions in an arbitrary number of dimensions.