Islamic Mathematics

Elizabeth Rogers

MATH 390: Islamic Mathematics

Supervised by Professor George Francis

University of Illinois at Urbana-Champaign

August 2008

0.1 Abstract

This project will provide a summary of the transmission of Greek mathematics through the Islamic world, the resulting development of algebra by Muhammad ibn Musa al-Khwarizmi, and the applications of Islamic algebra in modern mathematics through the formulation of the Fundamental Theorem of Algebra. In addition to this, I will attempt to address several cultural issues surrounding the development of algebra by Persian mathematicians and the transmission of Greek mathematics through Islamic mathematicians. These cultural issues include the following questions:

☐ Why was the geometry of Euclid transmitted verbatim while algebra was created and innovated by Muslim mathematicians?

☐ Why didn't the Persian mathematicians expand or invent new theorems or proofs, though they preserved the definition-theorem-proof model for geometry? In addition, why did the definition-theorem-proof model not carry over from Greek mathematics (such as geometry) to algebra?

☐ Why were most of the leading mathematicians, in this time period, Muslim? In addition, why were there no Jewish mathematicians until recently? Why were there no Orthodox or Arab Christian mathematicians?

0.2 Arabic Names and Transliteration

Arabic names are probably unfamiliar to many readers, so a note on how to read Arabic names may be helpful. A child of a Muslim family usually receives a first name (*'ism*), followed by the phrase "son of $\cdots$ " (*ibn* $\cdots$). For example, Thābit ibn Qurra is Thābit, son of Qurra. Genealogies can be combined; for example, Ibrāhīm ibn Sinān ibn Thābit ibn Qurra means that Ibrāhīm is the son of Sinān, grandson of Thābit, and great-grandson of Qurra. A name indicating the tribe or place of origin (*nisba*), such as al-Khwārizmī (or, from Khwārizm), may also be added, as well as a nickname or title, such as al-Rashīd ("the orthodox"). Later in life, Abū so-and-so (father of so-and-so) can be added to the name, such as Abū ʻAbdullāh (father of ʻAbdullāh). In the transliteration of Arabic into English, Arabic short vowels are denoted with regular English vowels, while Arabic long vowels are denoted with bars over the English vowel (ā). The following is a list of the full Arabic names of the Muslim mathematicians and rulers that will be discussed, with the correct transliteration into English. From this point onward, I will omit the vowel bars in these names.

Caliph Abū Jafar al-Ma‘mun ibn Harun (786-833 CE; r. 813-833 CE)
Al-Hajjāj ibn Yūsuf ibn Matar (c. 786-833 CE)
Abū ‘Abdullah Muhammad Ibn Mūsā Al-Khwārizmī (800-847 CE)
Thābit ibn Qurra al-Harrānī (836-901 CE)
Abū Kāmil Shujū‘ ibn Aslam ibn Muhammad ibn Shujā (c. 850-930 CE)
Abū Nasr al-Farabi (870-950 CE)
Ibrāhīm ibn Sinān ibn Thābit ibn Qurra (908-946 CE)
Caliph Abū al-Qāsim al-Muti ‘llāh al-Fadhl ibn Ja‘far al-Muqtadir (914-975 CE; r. 946-974 CE)
Emir ‘Adud al-Daula (936-983 CE; r. 950-983)
Mohammad Abū’l-Wafa (940-998 CE)
Abū Sahl Wayjan ibn Rustam al-Qūhī (c. 940-1000 CE)
Abū Bakr ibn Muhammad ibn al-Husayn al-Karajī (953-1029 CE)
Ibn Yahyā al-Maghribī al-Samaw‘al (1130-1180 CE)

Contents

Chapter 1

Algebra

1.1 Introduction

Islamic mathematics refers to the mathematical theories and practices that flourished in the parts of the world where Islam was the dominant religious and cultural influence. Along with transmissions of Greek mathematics, Muslim mathematicians in the Islamic Medieval Empire expanded on the Greek concepts of geometry, astronomy, medicine, and arithmetic. Muslim scholars also consolidated Greek and Indian mathematics to form the beginnings of modern algebra. The House of Wisdom (bayt al-hikma) was established by the Abbasid caliph al-Rashid, and flourished under the caliph al-Ma'mun. It was at the House of Wisdom that al-Khwarizmi and others translated Greek and Indian mathematical and scientific works. The historical development of algebra will be the focus of this first chapter.

1.2 Al-Khwarizmi

1.2.1 Biography

We begin with a discussion of al-Khwarizmi, the father of algebra. Abu 'Abdullah Muhammad Ibn Musa Al-Khwarizmi lived about 800-847 CE, but these dates are uncertain. The epithet "al-Khwarizmi" refers to his place of origin, Khwarizm or Khorezm, which is located south of the delta of the Amu Dar'ya River and the Aral Sea in central Asia. However, the historian al-Tabari adds the epithet "al-Qutrubbulli," indicating that al-Khwarizmi actually came from Qutrubull, near Baghdad between the Tigris and the Euphrates Rivers [06], 50. Other sources state that his “stock” comes from Khwarizm, so perhaps al-Khwarizmi’s ancestors, rather than himself, come from Khwarizm [09], 3. Another interesting epithet added by al-Tabari is “al-Majusi,” which would mean that al-Khwarizmi was an adherent of the Zoroastrian religion. However, al-Khwarizmi’s preface to his treatise on algebra shows beyond doubt that he was a devout Muslim; perhaps some of his ancestors or even al-Khwarizmi in his youth were Zoroastrian [06], 52.

Al-Khwarizmi grew up near Baghdad under the reign of Caliph al-Ma’mun (reign 813-833 CE), who was a great promoter of science. Al-Khwarizmi was offered a position at the Bayt al-Hikma (House of Wisdom) in Baghdad; most of his treatises are dedicated to the Caliph al-Ma’mun [06], 53.

1.2.2 Al-Khwarizmi’s Mathematical Contribution

Astronomy Most of al-Khwarizmi’s treatises are in the field of astronomy. He was one of the developers of the astrolabe and also wrote about a hundred astronomical tables. One of these, *Zij al-sindhind*, is the first Arab astronomical work to survive in its entirety [06], 55-6. He also wrote a geography text, *Kitab surat al-ard*, which listed the longitudes and latitudes of cities and localities. This was based on al-Ma’mun’s world map, on which al-Khwarizmi had worked, which was in turn based on Ptolemy’s *Geography.* However, al-Ma’mun’s world map was much more accurate than Ptolemy’s, especially in concerning the Islamic world [09], 9.

Calendar Another surviving work of al-Khwarizmi is his work on the Jewish calendar, which accurately describes the 19-year cycle, its seven months, and the rules for determining which day of the week the month of Tishri begins on. He also calculates the interval between the Jewish Era, or the creation of Adam, and the Selucdid era, which began October 1, 312 BC. Finally, al-Khwarizmi includes a method for determining the mean longitude of the sun and moon [06], 58.

Arithmetic In addition to his works on algebra, the treatises of al-Khwarizmi which have ensured his lasting fame are his works on arithmetic. His arithmetic treatise was possibly entitled *Kitab al-iam wa’l-tafriq bi-hisab al-Hind*, or *Book of Addition and Subtraction by the Method of Calculation of the Hindus* [06], 59-60. However, the original Arabic manuscript is now lost, and his text survives only in its Latin translation, which may have been done by Adelard of Bath in the 12th century. It was first published as *Algoritmi de numero indorum* by B. Boncompagni in 1857, and later published as *Mohammed ibn Musa Alchwarizmi's Algorithmus* by Kurt Vogel in 1963 [09], 9. This is the first known textbook written on the decimal system, and it is the first treatise to "systematically expound the use of the Arabic (or sometimes Hindu-Arabic) numerals 1-9, 0 and the place-value system" [06], 61. The introduction of the numeral 0 was most important; the "small circle is actually one of the world's greatest mathematical innovation" [06], 62. The symbol 0 was used for about 250 years in the Islamic world after its introduction by al-Khwarizmi before the Western world ever knew of it.

Modern numeral notation certainly has its roots in al-Khwarizmi and other Arab mathematicians; though influenced by Hindu numerals, al-Khwarizmi and his Arab successors introduced the full concept of ten numbers and the method of decimal notation [06], 63-5. Al-Khwarizmi introduced the zero, and his accounts of Hindu numerals were so accurate that he is probably responsible for the widespread belief that our system of numeration is Arabic. Though al-Khwarizmi never claimed originality regarding his number system, Latin translations of his work were widespread in Europe, and careless readers attributed both the book and the numeration to the author [05], 256. It is this association with numbers which led to the distortion of al-Khwarizmi's name to algorismi, which in turn led to the modern word algorithm.

1.2.3 Al-Khwarizmi's "Algebra"

al-jabr and al-muqabalah For the interests of this paper, the topic of most importance will be al-Khwarizmi's treatise *Kitab al-jabr wa'l-muqabalah*, or *The Book of Restoring and Balancing* [05], 256. The meanings of the words *al-jabr* and *al-muqabalah* are debated. *Al-jabr*, which comes to us in its form "algebra," probably meant something like "restoration" or "completion," referring to the transposition of subtracted terms to the other side of the equation or adding equal terms to both sides of the equation to eliminate negative terms [05], 257. *Al-muqabalah* probably means something like "restoration" or "balancing," referring to the cancellation of like terms on opposite sides of the equation, or reduction of positive terms by subtracting equal amounts from both sides of the equation [05], 257. Together, the two words *al-jabr wa'l-muqabalah* can mean the science of algebra. Al-Khwarizmi's treatise was the first book to use this title to designate algebra as a separate discipline.

It may be helpful to see examples of how al-Khwarizmi used these terms. He first poses the problem:

> I have divided ten into two portions. I have multiplied the one of the two portions by the other. After this I have multiplied one of the two by itself, and the product of the multiplication by itself is four times as much as that of one of the portions by the other. [09], 4

Al-Khwarizmi calls one of the portions "thing" and the other "ten minus thing." He multiplies by two, getting "ten things minus a square," and then obtains (in modern notation):

$$x^2 = 40x - 4x^2$$

He uses *al-jabr* to add $4x^2$ to both sides, which yields:

$$5x^2 = 40x$$

Al-Khwarizmi then gets

$$x^2 = 8x$$

from which he obtains

$$x = 8$$

(It is apparent to the modern reader that al-Khwarizmi does not allow x to equal 0.) On another page, al-Khwarizmi has the equation: [09], 4-5

$$50 + x^2 = 29 + 10x$$

He uses *al-muqabalah* to reduce both sides by 29 to obtain:

$$21 + x^2 = 10x$$

Origins of Algebra It is important to note that the origin of algebra does extend back to the ancient Egyptians and Babylonians, who had texts dealing with problems of arithmetic, algebra, and geometry as early as 2000 BC. In Diophantus' *Arithmetica*, several equations had already appeared. However, these equations were solved as parts of solutions to other problems and were not systematically treated. Al-Khwarizmi was the first to systematically study algebra. Though Diophantus' equations existed, al-Khwarizmi was probably not aware of them at the time he wrote his treatise; al-Khwarizmi did not know Greek, and there was no Arabic translation of *Arithmetica* at the time [06], 69-71. Al-Khwarizmi was probably more influenced by Hindu or local Syriac-Persian-Hebrew sources. However, none of these sources progressed as far as al-Khwarizmi; the few texts that do seem to have been written after *Kitab al-jabr wa'l-muqabalah* [06], 72. There seems to be no basis for the common Western view, present in *Algebra: Pure and Applied* by Aigli Papantonopoulou, which states that al-Khwarizmi is not an "original mathematician," since "there is little in his work that cannot be found in earlier Indian sources" [07], 438.

Algebraic Equations *Kitab al-jabr wa'l-muqabalah* has three sections, the first of which states that all linear and quadratic equations can be reduced to one of six types: [09], 5.

$$ax^2 = bx$$

$$ax^2 = b$$

$$ax = b$$

$$ax^2 + bx = c$$

$$ax^2 + c = bx$$

$$ax^2 = bx + c$$

He presents general solutions for all of these types. Looking at these six equations, it is apparent that al-Khwarizmi did not accept negative or zero coefficients [06], 74-5.

Al-Khwarizmi's treatment of mixed quadratic equations ("roots and numbers equal to squares," "squares and numbers equal to roots," and "roots and numbers equal to squares") is best seen with an example of the first type of mixed quadratic equations. In al-Khwarizmi's words:

> *Roots and Squares equal to numbers*
> For instance: one square and ten roots of the same amount to thirty-nine dirhems; that is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine?
> The solution is: you halve the number of roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of the roots, which is four. The remainder is three. This is the root of the square you thought for; the square itself is nine. [06], 77

In modern notation, the equation is

$$x^2 + 10x = 39$$

and al-Khwarizmi's solution is then

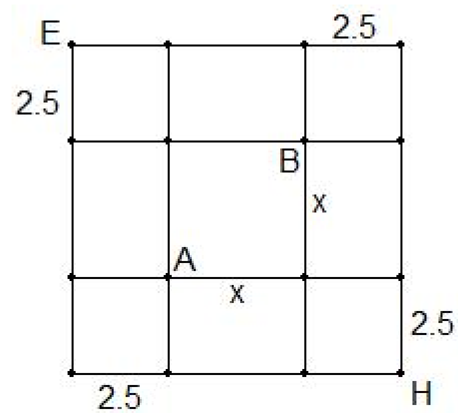
$$(x + 5)^2 = 39 + 25 = 64$$

$$x + 5 = \sqrt{64} = 8$$

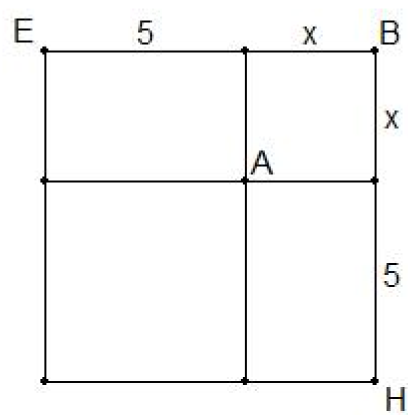
$$x = 8 - 5 = 3$$

$$x^2 = 9$$

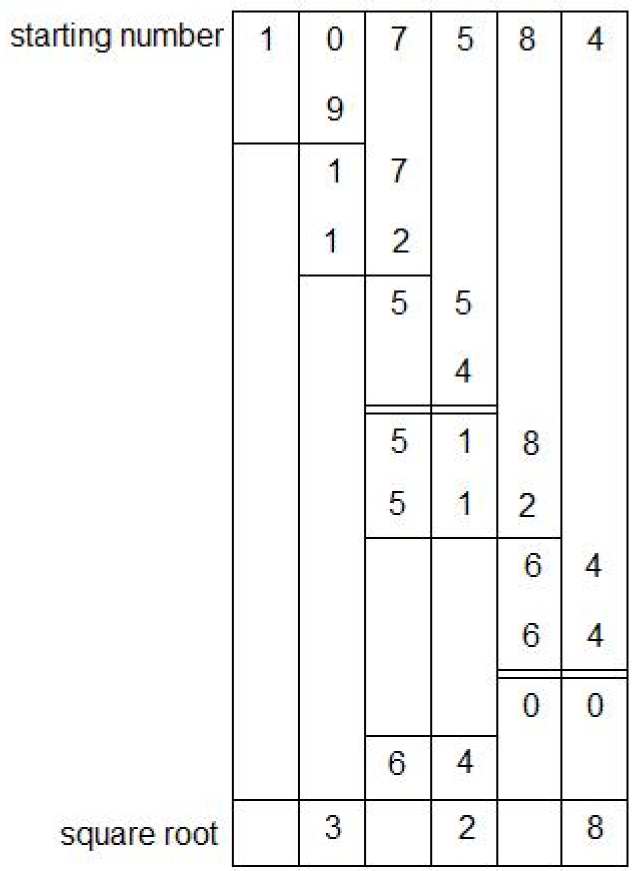
Al-Khwarizmi demonstrates this solution with a square AB, the side of which is the desired root x. On each of the four sides, he constructs rectangles, each having 2.5 as their width. So, the square together with the four rectangles is equal to 39. To complete the square EH, al-Khwarizmi adds four times the square of 2.5, or 25. So the area of the large square EH is 64, and its side is 8. Thus, the side x of the original square AB is $8 - 5 = 3$ [09], 8. (See the figure below.)



Al-Khwarizmi also presents a simpler, similar method which constructs rectangles of breadth 5 on two sides of the square AB. Then, the total area of the square EH is $x^2 + 10x + 25 = 39 + 25 = 64$, which yields the same result $x = 3$ or $x^2 = 9$ [09], 8. (See the figure below.)



Al-Khwarizmi also discusses methods of extracting the square root; this method may have been adapted from Hindu sources. It may be easiest to explain the method with an example. To find the square root of 107584, "vertical lines are drawn and numerals are partitioned into periods of two digits" [01], 53. The nearest root of 10 is 3, and so its square 9 is subtracted from 10. The 3 is written below everything as a part of the final square root. The 3 is doubled to make 6, which is contained twice in 17; $6 \cdot 2 = 12$, so 12 is subtracted from 17 which leaves 5. So 2 is written at the bottom as the next part of the final square root. Then, the square of 2 (which is 4) is then subtracted from 55, which leaves 51. So 518 is then divided by the double of 32 (which is 64), leaving 8. So $8 \cdot 64 = 512$ is subtracted from 518 leaving 6. The last figure then is 64, which is 8 squared, so the last figure of the final square root is 8. Therefore, the square root of 107584 is 328 [01], 53. (See the figure below.)



Mensuration Al-Khwarizmi's second chapter of *Algebra* is concerned with mensuration. It outlines rules for computing areas and volumes. The area of a circle can be found by multiplying half of the diameter by half of the circumference. To find the circumference, al-Khwarizmi provides three rules.

With the diameter d and the periphery p, and the approximate value of $\pi = p/d$:

$$p = 3\frac{1}{7}d, \text{ or } \pi \approx 3.1439$$
$$p = \sqrt{10d^2}, \text{ or } \pi \approx 3.1623$$
$$p = \frac{62832}{20000}d, \text{ or } \pi \approx 3.1416$$

The first rule was formulated by Archimedes and was also given in *Metrica* by Heron of Alexandria and the Hebrew treatise *Mishnat ha-Middot*. The second rule given can be found in the *Brahmasphutasiddhanta* of Brahmagupta. The third (equivalent to the accurate estimate $\pi \approx 3.1416$) is ascribed to "the astronomers" by al-Khwarizmi, which may refer to the Hindu astronomer Aryabhata; the same rule can be found in his *Aryabhatiya* [09], 6.

Al-Khwarizmi also states that for a rectangular triangle with sides a, $b\,c$, with a and b the "short" sides of the triangle, [09], 6

$$a^2 + b^2 = c^2$$

He provides a proof in the text; however, his proof is only valid for an equilateral triangle, when $a = b$. From this, it is apparent that al-Khwarizmi's main source cannot be a classical Greek treatise like Euclid's *Elements*. The Hebrew treatise *Mishnat ha-Middot* is closely connected with al-Khwarizmi's chapter on mensuration, which shows some type of direct dependence or a common source of both. If Solomon Gandz is correct that the author of the Hebrew treatise was Rabbi Nehemiah, who lived about 150 CE, al-Khwarizmi may have relied on the treatise or a Perian or Syrian translation of the text [09], 6-7. However, other authors are quick to point out that Solomon Gandz's conclusion is not well-supported, which leaves the date of origination of *Mishnat ha-Middot* open even until the period after al-Khwarizmi published his *Algebra*. Gad Sarfatti contends that *Mishnat ha-Middot* was not written until a later Islamic period, and may be an adaption of al-Khwarizmi's work [06], 72.

Legacies The last chapter of *Algebra* is the largest, and it is concerned mainly with legacies. It consists entirely of problems and solutions involving simple arithmetic and linear equations. These problems are not going to be discussed here as they use the same algebra already discussed and require an extensive knowledge of Islamic inheritance laws [09], 7.

Influence Al-Khwarizmi's *Algebra* was popular as soon as it was published, and Muslim mathematicians commented on it during al-Khwarizmi's lifetime. *Algebra* first became popular in the West when European scholars, such as Adelard of Bath (1120 AD) and Robert of Chester (1140 AD), began translating Arabic works into Latin. Leonardo of Pisa, also known as Fibonacci, includes many of the problems posed by al-Khwarizmi. However, these are most likely taken from Abu Kamil's texts, which use many of al-Khwarizmi's problems and solutions. William of Luna, another Italian mathematician, translated *Algebra* into Italian in the early 13th century; this translation was referenced by several scholars in the 16th century [06], 87. Al-Khwarizmi's treatise influence has reached the works of Johannes de Muris in the 14th century, Regiomontanus in the 15th century, and Adam Riese, Perez de Moya, Cardan, and Adrian Romain in the 16th century [06], 89-91. Even today, some elementary algebra teachers use al-Khwarizmi's suggestions, equations, and geometrical representations even without knowing their source. Mohini Mohamed summarizes al-Khwarizmi's lasting influence quite well, and so I leave the conclusion of his legacy to her:

> At the time of his death, the legacy that al-Khwarizmi left to the Islamic community included a way of representing numbers that led to a convenient method of computing, even with fractions; a science of algebra that would help settle problems of inheritance; and a world map that is more accurate than ever before.
>
> In the western world, mathematical science was more vitally influenced by al-Khwarizmi than by any other medieval writer. It is to al-Khwarizmi that we owe the widespread use of Arabic numerals. Positional notation in base 10, the free use of irrational numbers, and his introduction of algebra in the modern sense made him the principle figure in the history of Muslim mathematics. His introduction of Arabic numerals changed the content and character of mathematics and revolutionized the common practice of calculation in Medieval Europe. With the integration of Greek, Hindu, and perhaps Babylonian mathematics in his *Algebra*, this text is one of the best representations of the international character of Islamic Medieval Civilization. Among others, the words algebra, algorithm, cipher, and root survive as witnesses of the role played by al-Khwarizmi in the foundation and diffusion of

the science of calculation. [06], 91-2

1.3 Further Algebraic Developments

After al-Khwarizmi's innovative text *Algebra*, the development of algebra did not come to a stand-still. Several Muslim mathematicians are known for their work regarding algebraic developments.

Thabit ibn Qurra Thabit ibn Qurra (836-901 CE) followed al-Khwarizmi's general solutions; however, al-Khwarizmi presents his general proofs in conjunction with particular equations, whereas ibn Qurra presents his demonstrations in general. At this point, ibn Qurra had full access to Euclid's *Elements*, and freely used Euclid's theorems in his algebraic proofs. In the case $x^2 + px = q$, ibn Qurra correctly finds that $x = \sqrt{q + (\frac{p}{2})^2} - (\frac{p}{2})$ [03], 104-6. He follows his demonstrations with general proofs, following Euclid's examples of the definition-theorem-proof model.

Abu Kamil Abu Kamil (c. 850-930 CE) wrote a treatise titled *Algebra*, which was a commentary on al-Khwarizmi's work. His examples were later used by both the Muslim scholar al-Karaji in the late 10th century and the Italian Leonardo of Pisa, or Fibonacci, in the late 12th century. Many of his examples are taken from al-Khwarizmi, and like al-Khwarizmi's work, the entire work is written out, including numbers. Abu Kamil also discusses the geometrical proofs of equation solutions in terms of specific examples, like al-Khwarizmi, rather than using general proofs like ibn Qurra. Abu Kamil does go beyond the algebra of either ibn Qurra and al-Khwarizmi by providing rules for manipulating the following algebraic quantities:

$$(a \pm px)(b \pm qx) = ab \pm bpx \pm aqx + pqx^2$$
$$(a \pm px)(b \mp qx) = ab \pm bpx \mp aqx - pqx^2$$
$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$
$$\sqrt{a/b} = \sqrt{a}/\sqrt{b}$$
$$\sqrt{a} \pm \sqrt{b} = \sqrt{a + b \pm 2\sqrt{ab}}$$

Abu Kamil gives both algebraic and geometrical proofs for these equations [03], 108-10.

Al-Karaji Al-Karaji (953-1029 CE) tends to apply arithmetic to algebra, in contrast to Abu Kamil and ibn Qurra, both of whom apply geometry to algebra. Abu Bakr al-Karaji wrote *The Marvellous*, in which he develops the algebra of expressions using high powers of the unknown. He uses "root," "side," or "thing," to denote x, "mal" for x^2, "cube" for x^3, "mal mal" for x^4, "mal cube" for x^5, and so on. He creates each power of the unknown by multiplication by the previous elements; this was in innovation which allowed al-Karaji to treating equations such as $x^4 + 4x^3 - 6$ and $5x^6 - (2x^2 + 3)$ [03], 111-4.

1.3.1 al-Samaw'al

Ibn Yahya al-Maghribi al-Samaw'al (1130-1180 CE) was born in Baghdad. Though born to a Jewish family, he converted to Islam in 1163 after he had a dream telling him to do so. He was a popular medical doctor, and traveled around modern-day Iran to care for his patients, which included princes. His *The Shining Book on Calculation* gives rules for signs, creating the concepts of positive (excess) and negative (deficiency) numbers. He then gives rules for subtracting powers:

$$(-ax^n) - (-bx^n) = -(ax^n - bx^n), \qquad \text{if } a > b$$

$$(-ax^n) - (-bx^n) = +(bx^n - ax^n), \qquad \text{if } a < b$$

Al-Samaw'al sets out a chart to teach the reader how to multiply and divide simple expressions, such as "part of mal cube" or "mal mal cube", which are equal to $\frac{1}{x^5}$ and x^7. He also gives examples of the division of complex polynomials, which was a great development in algebra. His first example shows how to solve: [03], 115-7

$$\frac{20x^6 + 2x^5 + 58x^4 + 75x^3 + 125x^2 + 96x + 94 + 140x^{-1} + 50x^{-2} + 90x^{-3} + 20x^{-4}}{2x^3 + 5x + 5 + 10x^{-1}}$$

He creates a chart (see figure below) with the top row as the names of the orders in the natural sequence from left to right, and the row below that as the row of the answer, which begins empty and the is filled in as he proceeds. The rest of the chart is divided into horizontal bands, with two rows each [03], 115-7.

x^6	x^5	x^4	x^3	x^2	x^1	x^0	x^{-1}	x^{-2}	x^{-3}	x^{-4}
cc	*mc*	*mm*	*cube*	*māl*	*thing*	*unit*	*part*	*pm*	*pc*	*pmm*
			10	1	4	10	0	8	2	
20 2	2 0	58 5	75 5	125 10	96 0	94 0	140 0	50 0	90 0	20 0
	2 2	8 0	25 5	25 5	96 10	94	140	50	90	20
		8 2	20 0	20 5	86 5	94 10	140	50	90	20
			20 2	0 0	66 5	54 5	140 10	50	90	20
					16 2	4 0	40 5	50 5	90 10	20
						4 2	0 0	10 5	10 5	20 10

Al-Samaw'al begins by dividing 20cc by 2c to obtain 10c, and then subtracts (10c · divider) from the dividend. The old dividend is replaced by the remainder after the subtraction, and the divisor is copied to the right. Repeating this procedure, the leading 2 of the new dividend is divided by the 2 of the divisor, and the quotient, 1, is placed in the column to the right of the 10, in the answer row. Al-Samaw'al proceeds by repeating the procedure until he reaches his final result, which is

$$10x^3 + x^2 + 4x + 10 + 8x^{-2} + 2x^{-3}$$

Al-Samaw'al follows this example with several others, repeating the same procedure but allowing negative coefficients. His discovery of the procedure for long division was a significant achievement in Islamic algebra.

1.4 Sources

It is important to look at the sources al-Khwarizmi used, to determine the influences of Greek mathematics on his algebra. It is also important to discuss the modern sources of al-Khwarizmi, or how modern scholars know of al-Khwarizmi's work.

1.4.1 Al-Khwarizmi's Sources

It is natural to begin by looking of the sources used by al-Khwarizmi. There have been three theories advocated regarding the sources used by al-Khwarizmi at the beginning of algebra; these include the theories that he used classical Greek sources, or Hindu sources, or popular Syriac-Persian-Hebrew mathematical writings [09], 13.

According to Toomer, as discussed in B.L. van der Waerden's *A History of Algebra: From al-Khwarizmi to Emmy Noether*, both Hindu and Greek algebra had advanced well beyond the elementary stage of al-Khwarizmi's work. The proofs included throughout his work do not bear significant resemblances to known works from either culture. For example, his proofs of the methods of solution of quadratic equations drastically differ from the proofs found in Euclid's Elements [09], 14. Further, the surviving algebraic treatise of Greek culture, written by Diophantos, had developed towards symbolic representation, while al-Khwarizmi's treatise is rhetorical. In this respect, al-Khwarizmi's work is similar to that of Sanskrit algebraic works. For these reasons, it seems unlikely that al-Khwarizmi was influenced to any great extent by classical Greek mathematics.

Al-Khwarizmi did write a treatise on Hindu numerals, and two of his estimates for π are found in Hindu sources, supporting the theory that al-Khwarizmi's work was influenced by Hindu sources. Al-Khwarizmi further referenced his sources in his section on Mensuration in his algebra book: [09], 14

> The mathematicians, however, have two other rules for that. The one of them is: multiply the diameter with itself, then with ten, and then take the root of the product. The root gives the circumference.
>
> The other rule is used by the astronomers among them, and reads: multiply the diameter with sixty-two thousand eight hundred and thirty-two and then divide it by twenty thousand. The quotient gives the circumference.

The first rule ($p = \sqrt{10d^2}$, in modern notation) is found in Chapter XII of the *Brahmasphutasiddhanta* of Brahmagupta, supporting the theory that Al-

Khwarizmi was familiar with Hindu algebraic treatises [09], 15. Al-Khwarizmi attributed the second rule ($p = \frac{62832}{20000}d$, in modern notation) to "the astronomers," and the equation is found in the *Aryabhatiya* of the Hindu astronomer Aryabhata from the early sixth century AD. As al-Khwarizmi used both Persian and Hindu sources to compose his astronomical tables, it is plausible that he also derived his estimates of π from these sources.

The third theory contends that al-Khwarizmi's work was influenced by a local Syriac-Perisan-Hebrew popular tradition. This is supported by the close connection between the geometry of al-Khwarizmi and the Hebrew treatise *Mishnat ha-Middot*. This theory has also been supported by Solomon Gandz, the editor of *Mishnat ha-Middot*. He discusses his view of al-Khwarizmi as the "antagonist of Greek influence," stating that al-Khwarizmi never mentions his colleague, al-Hajjaj ibn Yusuf ibn Matar [09], 15. Al-Hajjaj devoted his life to the translation of Greek mathematical, philosophical, and scientific work into Arabic. However, al-Khwarizmi does not refer to Euclid and his geometry while writing his own geometrical treatise; further, al-Khwarizmi emphasizes his purpose of writing a practical algebraic treatise in contradiction to the Greek theoretical mathematics in the preface to his algebraic treatise. Because of this, Soloman Gandz contends: [09], 15

> Al-Khowarizmi [sic] appears to us not as a pupil of the Greeks but, to the contrary, as the antagonist of al-Hajjaj and the Greek school, as the representative of the native popular sciences. At the Academy of Baghdad [House of Wisdom] al-Khowarizmi represented rather the reaction against the introduction of Greek mathematics. His Algebra impresses us as a protest rather against the Euclid translation and against the whole trend of the reception of the Greek sciences.

It seems likely that though al-Khwarizmi may not have been influenced by Greek mathematics, a combination of the second and third theories may best describe the influences on al-Khwarizmi's algebra and geometry. Both Hindu sources and the popular mathematics of Syriac-Persian-Hebrew sources seem to be present in al-Khwarizmi's work, as seen through his use of *Brahmasphutasiddhanta*, *Aryabhatiya*, and *Mishnat ha-Middot*, and through the lack of similarities of al-Khwarizmi's algebra and geometry to Greek algebra and geometry.

1.4.2 Modern Sources of Al-Khwarizmi

Equally important in our discussion of al-Khwarizmi's work and his sources are the sources modern historians and mathematicians use to know his work. We know about medieval Islamic mathematics primarily through Arabic documents; the mathematical treatises of medieval Arab mathematicians can be found in libraries and private collections throughout the world. These collections are mainly found in the countries which were once part of the Islamic medieval world, but significant collections also exist in England, France, Germany, and Russia: all countries which were colonial powers in the Islamic world [04], 515-6.

Most of these treatises are prose compositions, but can include tables of numbers, some with hundreds of thousands of entries. These tables were computed mainly for astronomical purposes, and almost never include explanations of how the numbers or entries were computed. Physical artifacts also provide important sources of Islamic mathematics, such as mathematical and astronomical instruments. Examples of these artifact are three world maps in the form of circular disks. These allowed the users to find the direction of Mecca by rotating a ruler around the center of the disk [04], 516. The lasting prose treatises, tables, and instruments allow modern scholars to study medieval Islamic mathematics.

A good excerpt (translated to English) of Al-Khwarizmi's *Al-jabr...* and his treatise on Hindu numbers can be found in J. Lennart Berggren's chapter "Mathematics in Medieval Islam" in *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*, Princeton University Press, 2007.

Chapter 2

Geometry

Greek Sources After the late 8th century, Euclid's *The Elements* were translated into Arabic through the House of Wisdom in Baghdad. There are many Arabic editions of and commentaries on *The Elements*, which shows the influence Euclid had on Islamic mathematics, and especially on Islamic geometry [03], 72. Muslim mathematicians also had a great respect for Archimedes' *On the Sphere and Cylinder*. In his preface, Archimedes mentions his discovery of the area of a segment of a parabola; since his treatise on this particular subject was not translated into Arabic, Thabit ibn Qurra and his grandson Ibrahim ibn Sinan searched (with great success) for a proof of Archimedes' result. Thabit ibn Qurra actually translated or revised translations of all of the Archimedean works existing in medieval Arabic, including the text *The Heptagon in the Circle*, which Arabic sources attribute to Archimedes but does not exist in Greek [03], 72.

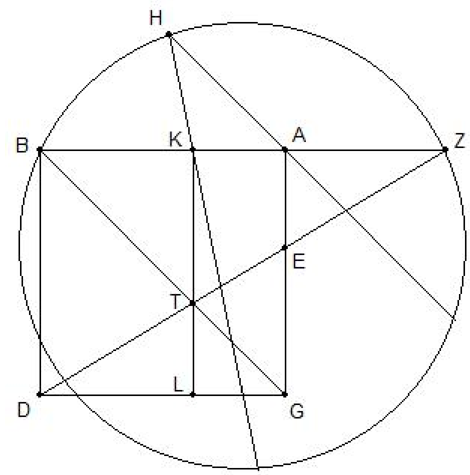
Another important work in Islamic geometry was Apollonios of Perga's *The Conics* from about 200 BC. Though *The Conics* contained eight chapters or books, only four exist in Greek and only seven in Arabic. These three Greek scholars: Euclid, Archimedes, and Apollonios, formed the basis of Islamic mathematics. Muslim mathematicians and translators are responsible for the preservation and transmission of these texts through the medieval period.

2.1 Abu Sahl

Abu Sahl Wayjan ibn al-Kuhi lived around 940-1000 CE. He was from Kuh, a mountainous area along the southern coast of the Caspian Sea in modern Iran. (Kuh is the Persian word for mountain.) Abu Saul worked in the Baghdad and is considered one of the greatest Muslim geometers in the 10th century. Though Abu Sahl worked on many treatises on geometry and astrology, his explanation of the construction of a regular heptagon best shows his innovation as a geometer and his contribution to Islamic mathematics by providing solutions to "impossible" problems within known mathematical theories [03], 77.

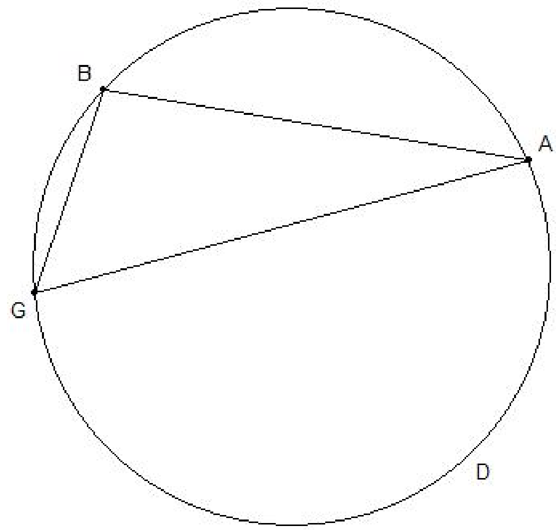
2.1.1 Regular Heptagon

Archimedes Archimedes' construction of the regular heptagon was unexplained and unique in Greek mathematics, and so served more as a proof of the existence of a regular heptagon than a construction of it [03], 78. Archimedes' regular heptagon followed this explanation: Begin with a square $ABDG$ and its diagonal BG. Archimedes then drew a line from D so it crossed BG at point T, the side AG at point E, and the extension of side BA at point Z. Then, the triangle AEZ has the same area as DTG. Next, Archimedes drew KTL parallel to AG. He proved that K and A divide the segment BZ so that the segments BK, KA, and AZ form a triangle and so that $BA \cdot BK = ZA^2$ and $KZ \cdot KA = KB^2$. Then, he forms triangle KHA so that $KH = KB$ and $AH = AZ$, and draws a circle BHZ. Finally, Archimedes proved that $\widehat{BH}$ is $\frac{1}{7}$ of the circumference of the circle [03], 78. (See figure below.)

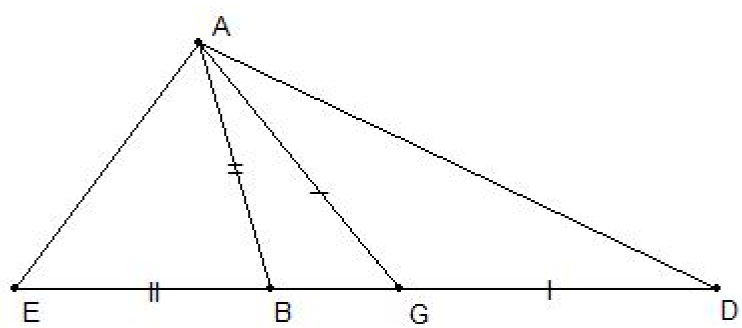


Construction by Reduction Abu Sahl analyzed the problem backwards, considering an already constructed heptagon and reasoning backwards. If his chain of reasoning can be reversed, then Abu Sahl has the proof of what is required starting from a given to the finished heptagon. The following analysis was provided in a treatise dedicated to King ‘Adud al-Daula, the Buwayhid ruler of modern-day Iraq and Iran under the Abassid caliph Al-Muti. His result showed that constructions which did not fit into any theories could be fitted into the theory of conic sections, a new development in geometrical thought.

First Reduction: From Heptagon to Triangle “Suppose that in the circle ABG we have succeeded in constructing the side $\widehat{BG}$ of a regular heptagon and that $\widehat{AB} = 2\widehat{BG}$. Then arc $\widehat{ABG} = 3\widehat{BG}$, and since $\widehat{BG}$ is 1/7th of the whole circumference, $\widehat{ADG} = 4\widehat{BG}$. According to VI, 33 of Euclid’s *Elements* angles of $\triangle(ABG)$ on the circumference are proportional to the arcs they subtend, and therefore $\angle B = 4\angle A$ while $\angle G = 2\angle A$. Thus, the construction is reduced to the problem of constructing a triangle whose angles are in the ratio 4:2:1.” [03], 79-80. (See figure below.)



Second Reduction: From Triangle to Division of Line Segment Now suppose ABG is a triangle so that $\angle B = 2\angle G = 4\angle A$. Extend BG in both directions to points D and E so that $DG = GA$ and $EB = BA$. Complete the triangle $\triangle(AED)$. Now, let $\angle A = \angle BAG$, $\angle B = \angle ABG$, and $\angle G = \angle BGA$. Notice that $\angle G$ is an exterior angle of the isosceles triangle ABD, where $AG = GD$. So, $\angle G = \angle DAG + \angle D = 2\angle D$. Since we know $\angle G = 2\angle A$, we now know that $\angle A = \angle D$. Now, notice that $\angle B$ is an exterior angle of the isosceles triangle ABE, so $\angle B = 2\angle BAE$. We already know that $\angle B = 2\angle G$, so $\angle BAE = \angle G$ [03], 80-1. (See figure below.)



Now that we know $\angle A = \angle D$, we know that $\triangle(ABG) \sim \triangle(DBA)$. Since we just established that $\angle BAE = \angle G$, we know that $\triangle(AEB) \sim \triangle(GEA)$.

So, $DB/BA = AB/BG$ and $GE/AE = AE/BE$. Thus,

$$BA^2 = DB \cdot BG \text{ and } EA^2 = GE \cdot EB.$$

Now, since $AB = BE$, $\angle E = \angle BAE = \angle G$, so we have $EA = AG = GD$. Thus, these equalities become

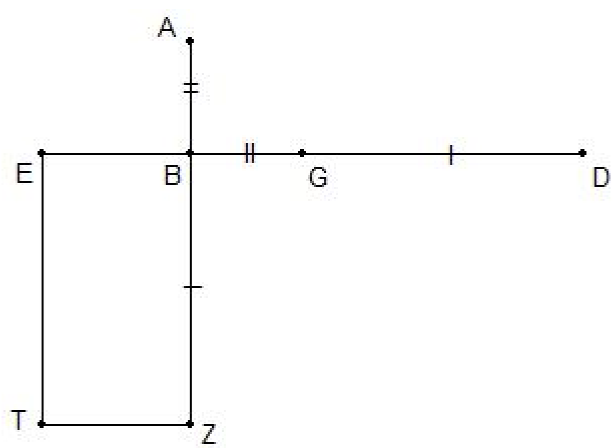
$$GE \cdot EB = GD^2 \text{ and } DB \cdot BG = BE^2.$$

So, Abu Sahl reduces the construction to the division of a straight line ED at two points B, G so that these equalities hold true [03], 80-1.

Third Reduction: From Divided Line Segment to Conic Sections Now suppose that we have a line segment ED with points B, G so that

$$GE \cdot EB = GD^2 \text{ and } DB \cdot BG = BE^2.$$

Draw a line ABZ perpendicular to ED with $AB = BG$ and $BZ = GD$, and complete the rectangle $BZTE$. Then we have $ZA \cdot AB = DB \cdot BG = BE^2$. (See figure below)



Since $AB = BG$ and $BE = TZ$, this leads to the equality $ZA \cdot BG = TZ^2$. This means that the point T lies on a parabola whose vertex is A and whose parameter is BG.

Further, we have $GE \cdot EB = GD^2$ and we know $GD = BZ = ET$, so we get the equality $GE \cdot EB = ET^2$, which says that the point T lies on a hyperbola with vertex B whose transverse side and parameter are both equal to the segment BG [03], 81-2.

Synthesis Thus, Abu Sahl reduced the divided line segment to the construction of two conic sections. Using synthesis, he puts the steps back in order to begin with conics and end with a regular heptagon: First, create two conics with intersection point T, which determines the lengths of ET and TZ. These produce the segments $GD = ET$ and $EB = TZ$ with the property that the line $EBGD$ is divided at B and G so that

$$GE \cdot EB = GD^2 \text{ and } DB \cdot BG = BE^2.$$

Thus, given BG, we can construct the line segment $EBGD$, then the $\triangle(ABG)$, and finally our regular heptagon.

Abu Sahl was the first to note that given a class of curves beyond a straight line and a circle (the class of conic sections), it was possible to construct in any circle the side of a regular heptagon. His proof showed that the construction of a regular heptagon belonged to an intermediate class of problems which had no previous solution and which required at times cubic curves. Abu Sahl limited both the level of difficulty of the problem and the means to solve it, placing the problem within the context of the known mathematical theory of conic sections [03], 82.

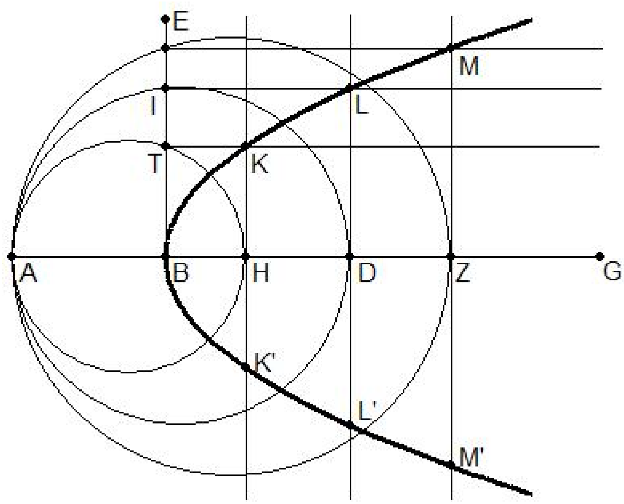
2.2 Ibrahim ibn Sinan

Ibrahim ibn Sinan (d. 946) is the grandson of Thabit ibn Qurra, the famous mathematician and translator of Archimedes. His treatment of the area of a segment of a parabola is the "simplest that has come down to us from the period prior to the Renaissance" [03], 87. He wrote that he invented the proof out of necessity, to save his family's scientific reputation after hearing accusations that his grandfather's method was "too long-winded" [03], 87. He also was concerned with methods and theories over particular problems, as seen in his treatise *On the Method of Analysis and Synthesis in Geometrical Problems*. His work *On Drawing the Three Conic Sections* is a discussion, with proofs, of how to draw the parabola and ellipse. He also gave three methods for drawing the hyperbola, which may be because of the interest in the hyperbola by instrument-makers.

On the Parabola Ibn Sinan describes the following method for drawing a parabola [03], 87-8. First, draw a line AG. Create a fixed segment AB on

AG and construct BE perpendicular to AB. On BG, pick as many points $H, D, Z, \ldots$ as you wish. Starting with the point H, create a semicircle with diameter AH, and let the perpendicular BE intersect it at T. Through T create a line parallel to AB and through H draw a line parallel to BE. These lines intersect at K.

Now, draw a semicircle with diameter AD, and let this intersect BE at I. Following the same procedure as above, draw a line through I parallel to AG and a line through D parallel to BE. Let these lines intersect at L. Follow the same construction method for the remaining points $Z, \ldots$ to obtain the corresponding intersection points $M, \ldots$. Then these points B, K, L, M, $\ldots$ lie on the parabola with vertex B, axis BG, and parameter AB. Create $K', L', M', \ldots$ on the extensions of the lines $KH, LD, MZ, \ldots$, respectively, so that $KH = HK', LD = DL', MZ = ZM', \ldots$. Then these points $K', L', M', \ldots$ also lie on the parabola. (See figure below.)



Ibn Sinan also proves that K is on the parabola. He assumes that the parabola does not pass through K, which means that it must pass through another point N on KH. Then, $NH^2 = AB \cdot BH$, by the property of the parabola. However, since TB is perpendicular to the diameter of the semicircle ATH, he points out that $TB^2 = AB \cdot BH$, by a rule from Euclid's *Elements*. Further, he has constructed $TBHK$ to be a parallelogram, so $TB = KH$. So, $KH^2 = TB^2 = AB \cdot GB = NH^2$ which means that $KH = NH$ and $K = N$, which contradicts his first assumption. Therefore,

K lies on the parabola. Ibn Sinan applies the same proof to $L, M, \ldots$ to prove the validity of his parabolic construction [03], 88. This method shows the ability of Ibn Sinan and Muslim mathematicians to construct a proof in the style of the Greeks (as a proof by contradiction originated with the Greek mathematicians), as well as their contributions to geometry by providing more concise geometrical constructions and proofs.

2.3 Geometrical Designs

One of the most impressive parts of Islamic culture has always been the elaborate geometrical artwork showcased in wood, tile, paintings, and mosaics. As geometers recognized this tradition, as well as the geometrical problems artists solved, they began to "justify the procedures and to see how far various methods could be pushed" [03], 89. The eighth book of Pappos of Alexandria's *Mathematical Collection* deals with instruments and machines of craft artisans, and includes an interesting section on geometrical constructions that can be created with only a straightedge and a compass with one fixed opening, sometimes referred to as a "rusty compass" [03], 89-90. This text, and especially the eighth book, was translated into Arabic and copied numerous times, pointing to its wide influence in the Islamic empire.

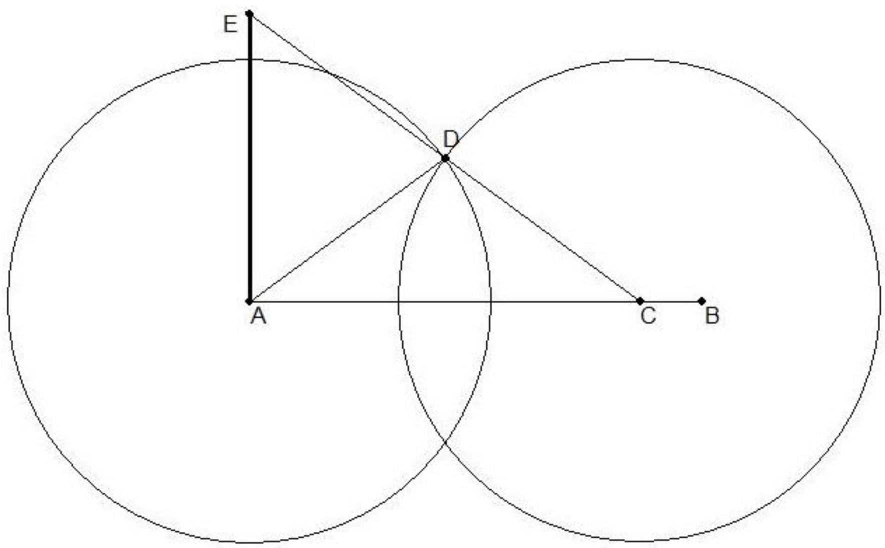
Abu Nasr al-Farabi Another text on geometrical constructions is one by Abu Nasr al-Farabi (870-950 CE). He taught philosophy in both Baghdad and Apello (in northern Syria), and was killed by highway robbers outside Damascus in 950 CE. He wrote a treatise called *A Book of Spiritual Crafts and Natural Secrets in the Details of Geometrical Figures*, which was later incorporated into Abu'l-Wafa's work *On Those Parts of Geometry Needed by Craftsmen*. Translations of several problems from this treatise are shown here, taken from the excerpts provided in [03], 90-2.

2.3.1 Problem 1

"To construct at the endpoint A of a segment AB a perpendicular to that segment, without prolonging the segment beyond A. [See figure below.]

Procedure. On AB mark off with the compass segment AC, and, with the same opening, draw circles centered at A and C, which meet at D. Extend CD beyond D to E so that $ED = DC$. Then $\angle CAE$ is a right angle.

Proof. The circle that passes through E, A, C has D as a center since $DC = DA = DE$. Thus EC is a diameter of that circle and therefore $\angle EAC$ is an angle in a semicircle and hence is a right angle" [03], 92.



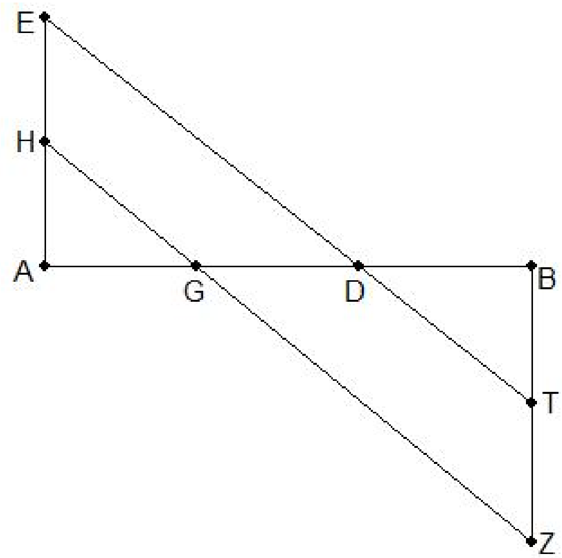
2.3.2 Problem 2

"To divide a line segment into any number of equal parts [– for example, three equal parts. (See figure below.)]

Procedure. Let it be required to divide the line segment AB into the [three] equal parts $AG = GD = DB$. At both endpoints erect perpendiculars AE, BZ in different directions and on them measure off equal segments $AH = HE = BT = TZ$. Join H to Z and E to T by straight lines which cut AB at G, D respectively. Then $AG = GD = DB$.

Proof. Indeed, AHG and BTD are two right triangles with equal angles at G and D (and therefore at H and T). In addition $HA = BT$. Thus the triangles are congruent and so $AG = BD$. Also the parallelism of HG and ED implies that the two triangles AHG and AED are similar, and thus

$DG/GA = EH/HA$. But, $EH = HA$ and so $DG = GA$" [03], 92-3. This proof by similarity is obvious from the picture below.



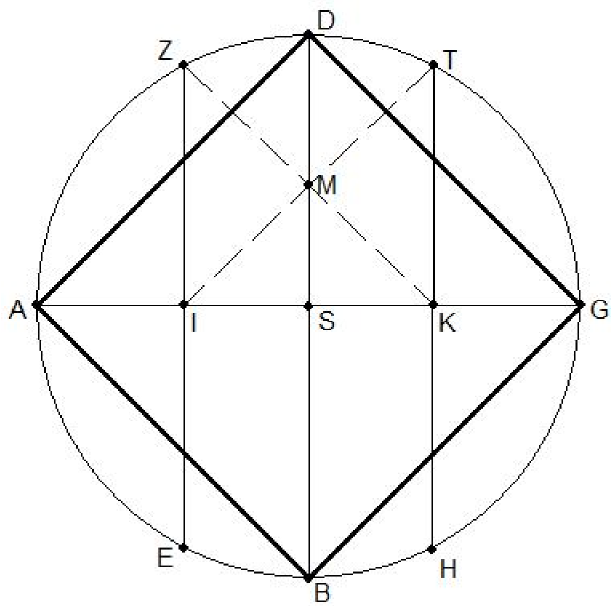
2.3.3 Problem 3

"To construct a square in a given circle. [See figure below.]

Procedure. Locate the center S and draw a diameter ASG. With compass opening equal to the radius, mark off arcs $\widehat{AZ}, \widehat{AE}, \widehat{GT}$, and $\widehat{GH}$ and draw the lines ZE and TH, which cut the diameter at I and K. Draw ZK and TI, which intersect at M, and then draw the diameter through S, M. Let it meet with the circle at D and B. Then $ADGB$ will be a square.

Proof. Since $\widehat{ZA} = \widehat{AE}$, the diameter GA bisects the arc $\widehat{ZE}$ and therefore GA is perpendicular to ZE, the chord of that arc. Similarly GA is perpendicular to TH, and so $\angle TKI$ and $\angle ZIK$ are right. Since TH and ZE are chords of equal arcs they are equal and therefore their halves, TK and ZI, are equal, and since they are also parallel (both being perpendicular to GA) the figure $TKIZ$ is a rectangle. Its diagonals ZK and TI therefore are equal and bisect each other, and so $MK = MI$, i.e. $\triangle(MKI)$ is isosceles. Since the equal chords ZE and TH are equidistant from the center (Euclid

III, 14), $KS = SI$, and so in the isosceles triangle MKI the line MS bisects the side KI and is therefore perpendicular to the side. Thus the diameter DB is perpendicular to the diameter GA and $ADGB$ is a square" [03], 93-4.



Conclusion Thus, the Islamic art of geometrical design led to an expansion of geometry proofs and constructions. The proofs provided here, excerpted from the treatise *On Those Parts of Geometry Needed by Craftsmen*, show that Muslim mathematicians were very concerned with the practical aspects of their work. In this, Abu l'Wafa, Abu Nasr al-Farabi, Ibn Sinan, and Abu Sahl were similar to al-Khwarizmi, who wrote his *Algebra* for the practical applications regarding legacies. Islamic geometry continued to provide proofs, and also constructed more concise proofs than those contained in Greek manuscripts. However, their geometry was restricted, with the exception of conic sections, to a flat surface.

Chapter 3

Culture

An understanding of the medieval Muslim mathematicians requires an understanding of the Arab and Islamic culture in which they lived and worked. Though an accurate and thorough study of a civilization cannot, of course, be completed in a few pages, this section will share a bit of the civilization so Islamic mathematics can be better understood.

3.1 Muhammad

Muhammad was born in Mecca around 570 CE. He was the son of Abdallah, a merchant, and Aminah. His father died a few months before his birth, and his mother passed away when he was six years old. Muhammad was raised by his paternal grandfather, Abd Al-Muttalib, and by his uncle Abu Talib. Muhammad became a merchant for a wealthy widow Khadijah, who promoted him to the position of managing her trading caravans. Khadijah later proposed to Muhammad, who accepted. Muhammad remained in a monogamous relationship until after Khadijah's death, and often spoke of his great love for her and her support of him. Khadijah was also the first contemporary Muslim, converting after Muhammad's first revelation. She is given credit for reassuring Muhammad of his role as a Prophet of God.

Muhammad began preaching Islam, which means "submission" [to God]. His main message was one of monotheism, though the Qur'an provided guidelines for worship of God in all areas of life. Islam is the completion of Christianity and Judaism, and Muhammad is the final and universal Prophet sent by

God to all peoples (the Seal of the Prophets). Mecca, Muhammad's home town where he began preaching Islam, had an economy greatly based on the pilgrims who visited the Kaba, which held hundreds of statues of local gods and goddesses. His message of monotheism, then, was seen as dangerous to the economy. Muhammad and his followers were forced to flee to Medina, who had offered him a job as a city mediator. This flight is called the hijra.

In Medina, Muhammad set up an Islamic city, where laws conformed to the regulations set forth in the Qu'ran. He and his followers also conquered Mecca a few years after the hijra; the capture of Mecca was a blood-less battle, though both Meccans and Medinians suffered heavy losses in battles before this final capture. Shortly after returning to Mecca, Muhammad passed away. He had not designated anyone to succeed him, which resulted in crisis and opposition political camps.

3.2 The Caliphate

After many political battles, Abu Bakr became the leader of the Muslim community. He was in power from 632-635 CE, and is the first of the four "Rightly-Guided Caliphs." Under Abu Bakr, the Muslim community began its expansion, entering Palestine, Transjordania, and Syria. After his death, Omar Al-Khattab became Caliph, ruling from 634-644 CE. Under his command, the Islamic empire expanded to include Egypt and parts of North Africa. They conquered all Roman dominions outside of Europe as well as the Sassanian Empire. When Omar Al-Khattab passed away, Othman ibn Affin became Caliph, ruling from 644-656 CE. His Caliphate was marked by further development and expansion of the Islamic world, which was continued by the fourth "Rightly-Guided Caliph," Ali ibn Abu Talib, who was Muhammad's cousin and son-in-law. He ruled from 656-661 CE.

The Umayyad Caliphate There were many political and religious disagreements between Ali and the Umayyad governor of Syria, Muawiyah ibn Abu Sufyan [01], 21-2. Muawiyah was named Caliph in 661 CE, and he ruled until 680 CE. The dynasty he founded, the Umayyad dynasty, ruled the Muslim world for ninety years. It was under Muawiyah that the capital of the Muslim empire became Damascus, and it is thanks to his rule that the Muslim society was consolidated. He developed a stable, well-organized state, which favored peaceful measures above all else. The next important caliph was Abd-al-Malik (ruled 685-705 CE) [01], 21-2. With Abd-al-Malik's reign, and the rule of his four sons who succeeded him, the Islamic world reached its greatest expansion. During this period, the language was officially declared to be Arabic by al-Walid (r. 705-15 CE), which consolidated a multitude of languages into a single, unifying language. From this, the translation of different scientific materials began [01], 22. However, in 747 CE, the Abbasids revolted against the Umayyad Caliphate.

The Abbasid Caliphate The Abbasid empire focused on an international identity. The capital was moved to Baghdad, which became *the* center for learning in the Muslim empire. Scholars from Syria, Iran, and Mesopotamia were brought to Baghdad in the late 8th century, which included Jewish and Christian scholars. The Caliph al-Mansur (r. 775-785 CE) began funding the study and translation of mathematical texts. It was in 766 CE that the *Sinhind*, the first mathematical treatise from India, was brought to Baghdad. This work is called the *Sinhind* in Arabic, but may refer to the *Brahmasphuta Siddhanta*, which was influential in the development of Algebra. This text was translated in 775 CE. Ptolemy's astrological *Tetrabiblos* was translated from Greek into Arabic in 780 CE [05], 254. The Abbasid Caliph Harun Al-Rashid (r. 786-809 CE) began a more rigorous translation of classical mathematics in Greek and Sanskrit to Arabic, and it was during his reign that a very few parts of Euclid's *Elements* were translated into Arabic. Al-Rashid also established the Bayt al-Hikma, or the House of Wisdom, in Baghdad. The Abassid Caliphate reigned until 1258 CE, when it was destroyed by the invading Mongols.

House of Wisdom The House of Wisdom flourished under the reign of Al-Rashid's son, al-Ma'mun. The House of Wisdom was primarily involved with the translation of philosophical and scientific works from Greek originals; Caliph al-Ma'mun is said to have had a dream in which Aristotle appeared to him, after which al-Ma'mun resolved to have Arabic translations of all the Greek works he could acquire [05], 255. It is during this period that Ptolemy's *Almagest* and a complete version of Euclid's *Elements* were translated into Arabic. According to tradition, Greek originals were brought to Baghdad by a delegation sent by Caliph al-Ma'mun to the country of Rome, referring to the Byzantine Empire. (The capital of the Byzantine

Empire, Constantinople, was known as "Second-Rome.") Greek manuscripts were obtained through treaties with the Byzantine Empire, with which the Islamic Empire had an uneasy peace. Among the famous mathematicians employed at the House of Wisdom was al-Khwarizmi. In addition to "compiling the oldest astronomical tables, al-Khwarizmi composed the oldest work on arithmetic and the oldest work on algebra. These were translated into Latin [in the 12th century] and used until the sixteenth century as the principal mathematical textbooks by European universities" [01], 23-4. His work also introduced algebra, both the mathematical subject and the word, and Arabic-Indian numerals to Europeans.

3.3 Muslims in Europe

Spain Around 750 CE, the Umayyad dynasty in Damascus was overthrown by the 'Abbasid family. Among those of the Umayyad family who escaped was Abd-al-Rahman; al-Rahman went to Spain and fought to maintain the Umayyad dynasty in the West, though it was destroyed in the East. He also made Cordova into a center of world culture; through his efforts and the efforts of his two successors, Cordova became the most cultured city in Europe. Al-Hakam (r. 961-976 CE), successor to Abd-al-Rahman III, was a great patron of scholars and learning. He "granted lavish subsidies to scholars and established twenty-seven free schools in the capital" [01], 24-5. Under his patronage, the University of Cordova became a place of pre-eminence among world educational institutions. Its students, both Christians and Muslims, came from not only Spain but also other parts of Europe, Africa, and Asia [01], 25.

Sicily The only other area Muslims held power in Europe was in Sicily. The Muslim conquest of Sicily began with periodic raids in 652 CE, but was not completed in 827 CE. Sicily was transformed into a province of the Muslim world with Palermo as its capital, under the rule of Muslim chieftains, for the next 189 years. "By being at the meeting point of two cultural areas, Sicily became a medium for transmitting ancient and medieval culture" [01], 25. Its population included Greek elements, Arab elements, and a group of scholars who used Latin. All three languages were used in official registers and royal charters, as well as in the general population of Palermo. The translation of Greek writings dealing with astronomy and mathematics occurred here, as well as being translated in Toledo in Spain. Sicily and Italy were very important the spread of Greek mathematics in the Islamic Empire and the spread of Greek and Islamic mathematics to greater Western Europe.

3.4 Mathematics and Culture

The mathematics which was absorbed by Muslim scholars came from three primary traditions: Greek mathematics, Hindu mathematics, and practitioners of mathematics. Greek mathematics includes the geometrical classics of Euclid, Apollonius, and Archimedes, as well as the numerical solutions of indeterminate problems in Diophantus' *Arithmetica*. It also includes the practical manuals of Heron. The second tradition, Hindu mathematics, includes their arithmetic system based on nine signs and a dot for an empty space, as well as their algebraic methods, an emerging trigonometry, methods in solid geometry, and solutions of problems in astronomy. The third tradition, "mathematics of practitioners," includes the practical mathematics of surveyors, builders, artisans in geometrical design, merchants, and tax and treasury officials. This mathematics was part of an oral tradition which "transcended ethnic divisions and was a common heritage of many of the lands incorporated into the Islamic world" [04], 516.

Medieval Islamic mathematics not only reflected these three sources but also gave a primary importance to the Muslim society that sustained it. This can be seen in al-Khwarizmi's application of his algebra to the Islamic inheritance laws [04], 518. Islamic mathematics in the eighth through the thirteenth centuries was marked with a steady development in conic theory in geometry, methods and theories of solving general geometrical problems, treatment and definitions of irrational magnitudes, trigonometry, algebra, and the geometrical analysis of algebra. One important aspect of Islamic mathematics, in contrast to Greek mathematics, is the close relationship between theory and practice. For example, mathematical works discuss solutions to problems which arise when creating modules for use in Islamic tessellations, relating to the Islamic architectural decorative designs [04], 519. Mathematicians took into account the objections of artisans to their theoretical methods, and artisans also learned to understand the differences between exact and approximate methods.

Another example is the mathematical instrument, the astrolabe. It used "the circle-preserving property of stereographic projection to create an analog computer to solve problems of spherical astronomy and trigonometry" [04], 519. This is a good example of the intersections of mathematical traditions and Islamic culture, as the astrolabe was a Greek invention but Muslims added circles indicating azimuths on the horizon, which proved useful in determining the direction of Mecca. However, the construction of these circles was not just for religious purposes, but instead stimulated geometrical investigations [04], 519. Mathematics blended together with Islamic culture in a way that is quite distinct from any of the three mathematical traditions from which Muslim mathematicians acquired their knowledge.

3.5 Summary

The *Qur'an* and *hadith*, stories about the life of Muhammad, indicate that the religion of Islam aspired to expand to a major political system as well as assuming its role as a universal faith. For example, Muhammad sent emissaries to the King of Ethiopia and the rules of Iran and the Byzantine Empire, inviting them to convert to Islam. It is this aspiration which led to the expansion of Muslim economic, political, and religious influence over such a large territory in so short a time. Islam was meant to bring the world under "one system of religion, one form of government, and one way of life" [01], 27. However, the Islamic Empire was overtaken by the Mongols, and then by the Ottoman Turkish Empire, and then by colonial Western powers. Today, the parts of the world that were once part of the Islamic Empire still identify with the Islamic religion and guides for legal structures; however, each country implements these differently, and there is no longer the same rapid development in mathematics in the Eastern world that was seen in the Islamic medieval period.

Chapter 4

Conclusions

Throughout my paper, I have touched on all of the preliminary questions with which I began my study; however, it is convenient to discuss these questions and my conclusions in a separate section. The first observation that needs to be made is the surprising amount of Islamic mathematics which is relevant to contemporary studies, but is not part of any mathematical course. This is vastly apparent even with my three questions that I set out to answer; many of my presuppositions and assumptions were wrong, which made the questions themselves invalid. It is astounding how much historical mathematics is omitted from mathematical courses, which could greatly benefit from the inclusion of even a small amount of this material. In many cases, my understanding of modern mathematics, especially algebra, was enhanced by my study of Islamic mathematics; the historical overview gave insights to modern notation and modern methods of problem-solving.

4.1 On Transmission

I began my study hoping to study the transmission of Greek mathematics through the Islamic Empire to Western Europe. Though I have discussed this throughout my paper, it seems fitting to include a brief summary of the transmission. Greek mathematical and philosophical treatises were brought to the Byzantine Empire before the advent of Islam and the Islamic Empire. In the early stages of the Islamic world, there was a great deal of political unrest, both inside the Empire and with outside forces. As the Islamic Empire expanded, it faced the difficulty of uniting all of the people of such a large territory. It really was not until the Abassid Caliphate when the Empire began to stabilize, when it had reached the peak of its expansion, that the Islamic world was able to concentrate resources on centers of learning and culture rather than on internal cohesion and external expansion. The Abassid Caliph al-Rashid founded the House of Wisdom in Baghdad, where he summoned scholars of all nationalities and religions. These scholars were engaged in translating and writing treatises on mathematical and philosophical topics. This institute formed the basis of Islamic mathematical development. It was under the Abassid Caliph al-Ma'mun that the House of Baghdad began to flourish. Al-Ma'mun actively sought out classic Greek treatises for translation into Arabic, and treaties with the Byzantine Empire brought most of the Greek mathematical and philosophical works to Baghdad for translation.

These Greek translation and original Arab treatises reached Western Europe in a number of ways. First, al-Andalus, the Arab name for the Iberian Peninsula, was ruled by the Umayyid Caliphate (in exhile). The Umayyid rulers established many Universities and made Cordova into an important cultured city. Though the Umayyids and Abassids had a very uneasy peace, Abassid caliphs seem to have been willing to send copies of their Greek and Arabic texts to the universities at Cordova. Al-Andalus was thus a resource for Western contacts to receive Greek and Arab mathematical treatises.

The second route of transmission was through Italy. Not only was a Muslim presence established through trade routes in Italy and to the Muslim world, but Muslims had a firm presence in Sicily, where scholars of Greek, Arabic, and Latin came together to translate and study mathematical texts. Sicily had a good relationship with Baghdad, and thus Italy became an vital link in the transmission of Greek and Islamic mathematics to Western Europe.

The last route of transmission was through trade routes and Western travelers to the Arab world. As already stated, some of these trade routes linked Italy, and scholars like Fibonacci, with Islamic mathematics. Other travel and trade routes, however, linked the rest of Western Europe to the Islamic Empire. Travelers and merchants were able to access and buy copies of Greek and Arab treatises translated to or written in Arabic. These texts were then brought back to Europe, where they were distributed to university scholars or monarchs. And so, over centuries, Greek mathematics was transmitted through the Islamic Empire to Western Europe.

4.2 On Geometry

I began this paper with the question: "Why was the geometry of Euclid transmitted verbatim while algebra was created and innovated by Muslim mathematicians? In other words, why was geometry not developed while algebra was both created and developed?"

As my research shows, the question itself is not correct. Geometry was developed and added to by Muslim mathematicians, though I began my study without knowing this. Still, geometry only developed so far, and usually developed in conjunction with algebra, so the topic remains one of interest and can still be discussed.

As I have discussed, Euclid's *The Elements*, Archimedes' *On the Sphere and Cylinder*, and Apollonios of Perga's *The Conics* were of utmost importance to Muslim geometers. The Muslim mathematicians can be credited with transmitting Euclid's geometry verbatim, as they were very conscientious about their translations and the dispersion of their translations meant that a very faithful copy Euclid's *Elements* was transmitted to Western Europe. The translations of Archimedes and Apollonios show the same concern for the exactness of the translation; the Muslim mathematicians were successful in transmitting Euclid, Archimedes, and Apollonios to Western Europe in verbatim forms. However, Muslim mathematicians were also successful in developing geometry.

Abu Sahl's explanation of the construction of a regular heptagon shows his innovation as a geometer and his contribution to Islamic mathematics by providing solutions to "impossible" problems within known mathematical theories. Though Archimedes provided a construction of a regular heptagon inside a circle, his construction was not proven and so was more of a proof of the existence of a regular heptagon inside a circle rather than a valid construction. Abu Sahl was able to construct and prove the 'impossible" by reducing the problem to the construction of two conic sections. By doing this, Abu Sahl showed that the construction of a regular heptagon belonged to an intermediate class of problems which required at times the use of cubic curves. Abu Sahl was able to reduce a problem into a method which was already accepted, that of conic surfaces, to provide a general solution for the regular heptagon problem. Further, the geometry of Ibrahim ibn Sinan shows the Muslim importance of concise proofs and constructions, and he improved upon the methods of the Greeks in drawing the parabola, ellipse, and hyperbola. The extension of geometry into practical spheres was also a development which was not seen in Greek mathematics. The existence of many treatises on the problems facing artisans as they constructed their geometrical artwork shows the intersection between theory and practice in Islamic geometry, and in all Islamic mathematics.

Geometry may not have developed to our modern form, or to 3-dimensional geometry beyond conic sections because of the environment in which Muslim geometers worked. In Islam, it is prohibited to make likenesses of people, and so Muslim artists developed geometrical design over attempting to draw people and objects in 3-dimensions. It is this drawing in 3-d which may have sparked the Western European geometrical development; therefore, the Islamic culture in the House of Wisdom precluded the further development of geometry past the point already discussed.

4.3 On Theorems and Proofs

Another question with which I began my study was the following: "Why didn't the Persian mathematicians expand or invent new theorems or proofs, though they preserved the definition-theorem-proof model for geometry? In addition, why did the definition-theorem-proof model not carry over from Greek mathematics (such as geometry) to algebra?"

This question was also misinformed from the beginning. Though al-Khwarizmi did not adopt Euclid's "definition-theorem-proof" model, later algebraic treatises do so. Al-Khwarizmi still recognizes the importance of proving all assertions, though he did not structure his theorem-proofs in exactly the same way that Euclid's *Elements* or other Greek mathematical treatises do. Still, the topic can be discussed even if the question was originally misstated.

Al-Khwarizmi did not, as I had previously assumed, have access to most Greek mathematical treatises. In fact, al-Khwarizmi only had access to a very few parts of Euclid's *Elements*. His college Thibet ibn Qurra was engaged in the translation of many Greek treatises during and after al-Khwarizmi's publication of *Algebra*, so al-Khwarizmi must have relied on Hindu and local

Syriac-Persian sources for his studies. Al-Khwarizmi did not have access to Greek manuscripts, as seen in his proof of $a^2 + b^2 = c^2$ which is only valid in the case when $a = b$. Euclid's *Elements* provides a more rigorous proof which holds for all cases; if al-Khwarizmi had access to this text, he surely would have recognized this and included Euclid's proof, as he included the proofs from Hindu texts of other theorems, such as how to take a square root, when the Hindu proofs surpassed his own. In this case, then, it is an easy answer regarding why the definition-theorem-proof model was not carried over from Euclid to algebra; the founder of algebra was not aware of the existence of the definition-theorem-proof model.

The study of Islamic mathematics, both geometry and algebra, have shown that the definition-theorem-proof model was commonly used after the Greek treatises, and therefore the Greek definition-theorem-proof-model, were available to Muslim mathematics. For example, this can be seen in the proofs of Ibn Sinan (discussed in an earlier chapter), who followed this model when discussing the area of a segment of a parabola, or in the discussion of $x^3+mx = n$ by Umar al-Khayyami (1048-1122 CE), which includes a theorem, solution, and proof. It is clear that though it was not considered necessary, most Muslim mathematical treatises included some form of the definition-theorem-proof model that they admired so much in Euclid's *Elements* and other Greek texts.

4.4 On Religion

The final question I asked before I began this study was the following: "Why were most of the leading mathematicians, in this time period, Muslim? Including, why were there no Jewish mathematicians? Why were there no Orthodox or Arab Christian mathematicians?"

This question, too, is flawed; there were Jewish and Christian mathematicians at the House of Wisdom in Baghdad. However, they were employed mainly as translators, as they knew either Hebrew and Arabic or Greek and Arabic. The discussion of this question requires a brief look at the status of Jews and Christians within the Islamic Empire.

The *Qur'an* accords Jews and Christians the status of "People of the Book."

Under Islamic law, then, Jews and Christians are given protected status as *"dhimmi"*. As dhimmi, Jews and Christians are given the freedom to practice their own religion, with the condition that they may not attempt to convert Muslims to their religion, as long as they pay a special tax to the Islamic government. Though this afforded them a protective status, Jews and Christians were thus second-rate citizens; they did not have equal benefits under the law or in court. Further, many Muslims and Islamic governments wished for their second-class status to be reflected in their standard of living and their political and societal prominence.

This may help account for the position of Jewish and Christian mathematicians. Though employed as translators by al-Rashid and al-Ma'mun, these scholars were not seen as equals of Muslim scholars at the House of Wisdom. The formation of Islamic theology was also happening during this period, and many Jews and Christians gained prominence from their positions as religious authorities; they were asked to account for their religion before the Caliph, Islamic theologians, and Islamic legal analysts. However, it seem that it was not recognized that Arab Jews and Arab Christians could be experts in intellectual fields besides religion; at least, it was thought that Muslim scholars could perform just as well in all intellectual fields besides religion, where the caliphs were fair enough to let Jews and Christians speak for themselves. This bias may be recognized in the financial support given by the Caliph to specific Muslim mathematicians to allow them to write original treatises. The caliphs may have been willing to pay Arab Jews and Arab Christians for their translation works, but seemed to favor Muslim mathematicians when requesting original mathematical treatises.

Chapter 5

Appendix I

5.1 Introduction

As a study of the transmission of Greek mathematics through the Muslim world, it is fitting to attach a brief look at the evolution of Algebra to contemporary methods, culminating in the Fundamental Theorem of Algebra. Tracing the evolution of algebra through the Fundamental Theorem of Algebra shows the influence Arab mathematicians have had on modern algebra.

5.2 Algebra in the European Middle Ages

5.2.1 Leonardo of Pisa, or Fibonacci

The first European algebraic advances occurred in the "early Renaissance," in the 13th century, beginning with Leonardo of Pisa, or Fibonacci. Leonardo was lived circa 1180 - 1240 CE. Though born in Pisa, Leonardo was raised in Bugia, where he learned about commerce and arithmetic. After traveling to Egypt, Syria, and Provenece, and learning methods of calculation. He concluded that the decimal positional system (the Indian numbering system) was superior to all other numbering systems [09], 36. He returned to Pisa, studied Euclid's *Elements*, and wrote *Book of the Abacus* with information about arithmetic, algebra, and geometry. It was in this book that Leonardo introduced the famous Fibonacci sequence:

$$1 + 1 + 2 + 3 + 5 + 8 + 13 + ...$$

where

$$a_{n+1} = a_n + a_{n-1}$$

He also wrote *Flower* and *Book of Squares*, in which he solved the cubic equation

$$x^3 + 2x^2 + 10x = 20$$

and found a rational solution of the generalized system

$$x^2 + a = u^2$$

$$x^2 - a = v^2$$

Leonardo notes that for this system to be solvable over the integers, with x, u, v a triple of pairwise coprime integers, a must be of the form $4kl(k+l)(k-l)$. He calls this a congruum, which is four times the area of a right triangle with legs $2kl$ and $k^2 - l^2$. Leonardo points out that a congruum cannot be a square, which implies Fermat's Last Theorem for $n = 4$, or the unsolvability of the equation

$$x^4 + y^4 = z^4$$

So, Leonardo formulated Fermat's Last Theorem for the case $n = 4$ about four hundred years before Fermat. However, Leonardo's proposed proof does contain an error [02], 56-7.

5.2.2 Algebraic Symbolism

For almost 300 years, no European mathematician emerged who could understand or expand upon Leonardo of Pisa's work. The second half of the 15th century, or the Renaissance, is when a revival of algebraic investigations began. Two events, the fall of Constantinople in 1453 and the invention of the printing press, contributed to this revival.

German mathematicians were helpful in the development of algebraic symbolism. Johannes Widman (ca. 1462-1498 CE) first introduced our signs of $+$ and $-$ in his book *A Quick and Beautiful Method of Calculation for all Merchants*. Adam Reis (1492-1559 CE) then wrote *Coss*, which used the following symbols: [02], 64

x^0	$\varnothing$	"number"
x^1	r	"root, thing"
x^2	z	"square"
x^3	c	"cube"
x^4	zz	"square-square"
x^5	β	"deaf solid"
x^6	zc	"square-cube"
x^7	$bi\beta$	"second deaf solid"
x^8	zzz	"square-square-square"
x^9	cc	"cube-cube"

German mathematicians clearly recognized a need for uniform symbolic notations.

Nicolas Chuquet (died ca. 1500) introduced the zeroth power for the unknown and negative powers of the unknown. He would write 12^3 for $12x^3$, and so an equation we may write as "$5x^3 \cdot 3x^{-1} = 15x^2$," Chuquet would write as "5^3 multiplied by $3^{1 \cdot \tilde{m}}$ yields 15^2" [02], 65.

The development of a uniform algebraic notation contributed to the first real advances in European algebra, which were connected with the solution of cubic and quartic equations.

5.3 European Algebraic Achievements

During the Renaissance (the 15th and 16th centuries), the countries of Italy, Spain, France, and England led the resurgence of art, science, and literature. The 16th century began the age of European algebra with the solution of cubic and quartic equations.

5.3.1 Cubic Equations

Scipione del Ferro (1456-1526 CE) solved the equation

$$x^3 + px = q \qquad p, q > 0$$

but kept his results secret; this was common at the time, as "the owner of a method could challenge his rival to a scientific duel and set him problems solvable by the method the rival was ignorant of. Victory in such a 'tournament' brought one fame and placed one at an advantage when it came to filling a desirable position" [02], 68. Del Ferro only passed his method to his student, Fiore, who challenged Niccolo Tartaglia (ca. 1499-1557 CE) to a duel.

Niccolo Tartaglia's story is interesting. He was born into a poor family in Brescia, and his father passed away when Tartaglia was only six years old. When the French sacked Brescia in 1512, Tartaglia was wounded in the jaw and larynx; his mother treated him with home remedies, since they were too poor to consult a doctor. The name "Tartaglia" is actually a nickname which means "stammerer"[02], 68. In spite of the challenges Tartaglia faced in his youth, he gained a great knowledge of mathematics and mechanics, and wrote an impressive treatise *The New Science*. After many attempts, Tartaglia discovered the solution for the above equation the very night before the duel with Fiore! Tartaglia was therefore able to solve all of Fiore's problems, whereas Fiore could not solve the mechanics problems Tartaglia asked him. A few days after the duel, Tartaglia also solved the equation: [09], 55

$$x^3 = px + q \qquad p, q > 0$$

Girolamo Cardano (died 1576 CE) was a "true Renaissance figure and embodied the good and bad characteristics of that period" [02], 69. After finding out that Tartaglia knew the secret solution to the cubic equation, he began a sneaky campaign to find out the answer himself. He invited Tartaglia to Milan from a famous man who "happened" to be out of town when Tartaglia arrived. Tartaglia then accepted Cardano's "hospitality" and, after Cardano swore secrecy, Tartaglia revealed his solution to Cardano. Six years later, in 1545, Cardano published *Great Art, or The Rules of Algebra*, which included Tartaglia's solutions to the cubic equations as well as Luigi Ferrari's solution of the quartic equation. Though Cardano did refer to del Ferro and Tartaglia in his first chapter, he received the credit for the solutions, which are now named for him [02], 70.

Cardano did progress a bit further than Tartaglia in noting that the solution for x and y such that

$$x + y = 10 \qquad xy = 40$$

was satisfied by

$$5 + \sqrt{-15}$$

and

$$5 - \sqrt{-15}$$

if

$$\sqrt{-15} \cdot \sqrt{-15} = (-15)$$

However, he did not attempt to solve the cubic equation using expressions of the form: [02], 71

$$\sqrt{m}, \quad m<0$$

5.3.2 Complex Numbers and Literal Calculus

Little is known about Rafael Bombelli (ca. 1526-1573 CE) other than the fact that he lived in Bologna. His *Algebra* was influenced by the Greek Diophantus. Bombelli included 143 problems with solutions from Diophantus; *Algebra* introduced these problems and methods to European mathematicians. In *Algebra*, Bombelli introduces integral powers of rational numbers, irrational magnitudes such as square and cube roots, and introduces complex numbers through a multiplication table. He calls $+\sqrt{-1}$ "piu di meno," or "plus from minus," and $-\sqrt{-1}$ "meno di meno," or "minus from minus" [02], 72. Bombelli defines the multiplication of these numbers, and considers other arithmetic operations such as addition of $a\sqrt{-1} \pm b\sqrt{-1}$ and raising $a + b\sqrt{-1}$ to the second, third, and increasing powers [09], 60-1. Bombelli was the first European who used Diophantus' algebraic methods; he followed Diophantus' method in Diophantus' introduction of negative numbers in his own introduction of complex numbers. Bombelli then used these to solve algebraic equations [02], 73. Bombelli's complex number-systems were written in polar form in the 18th century, and later analyzed in the 19th century. Gauss's construction of the arithmetic of complex numbers promoted their adoption, since Bombelli's complex numbers were now "genuine numbers" [02], 74.

Francois Viète was born in Fontenay-le-Comte, and did most of his work in mathematics while at the court of kings Henry III and Henry IV in Paris. His contribution to mathematics cannot be understated; de Thou, a French historian and statesman, wrote in 1625:

> Francois Viète, a native of Fontenay in Poitou, was a man of such immense genius and of such profundity of thought that he managed to reveal the innermost secrets of the most arcane sciences and easily managed to do all that human perspicacity is capable of. But of all the different studies that forever occupied his great and unwearied mind, the one he primarily applied his proficiency to was mathematics. So great was his mathematical distinction that all that the ancients had invented in this discipline, all that we missed as a result of the ravages of time that annihilated their creations, all these he reinvented, reintroduced, and enriched with much that was new. He thought so persistently that he would often spend three successive days in his study without food or sleep, except that from time to time he would rest his head on his arm for a brief spell of sleep to keep up his strength... [02], 75

Viète attempted to create a new science combining the geometry of the "ancients" with the ease of operations in algebra. In his *An Introduction to the Art of Analysis*, Viète introduced the language of formulas into mathematics; in this way, he created a literal calculus. He used literal notations of both parameters and unknowns, which made it possible to write equations and identities in a general form. Mathematical formulas are not just "a compact language for recording theorems... What counts is that we can carry out operations on formulas in a purely mechanical manner and obtain in this way new formulas and relations" [02], 77. We can obtain these new formulas by observing the rules of substitution, removing parenthesis, and reduction of similar terms. Literal calculus then "relieves the imagination," according to Leibniz, since mechanical computations replace some reasoning by using literal calculus [02], 77.

Viète denotes unknown magnitudes with vowels A, E, I, O, and U and known magnitudes with consonants B, C, D, etc. He adopts the symbols $+$ and $-$ for addition and subtraction, introduces the symbol $=$ for the absolute value of the difference of two numbers (or, $|B-C|$ in modern notation). Viète also uses the word "in" for multiplication and "applicare" for division. Further, Viète introduced the rules: [09], 64-5

$$B-(C\pm D)=B-C\mp D; B\cdot(C\pm D)=B\cdot C\pm B\cdot D$$

shown in modern notation, as well as operations on fractions (also shown in modern notation):

$$\frac{B}{D} + Z = \frac{B + Z \cdot D}{D}$$

Viète's next treatise, *Ad logisticam speciosam notae priores*, introduces some of the most important algebraic formulas, such as: [02], 79

$$(A + B)^n = A^n \pm nA^{n-1}B + ... \pm B^n, \qquad n = 2, 3, 4, 5$$

$$A^n + B^n = (A + B) \cdot (A^{n-1} - A^{n-2}B + ... \pm B^{n-}), \qquad n = 3, 5$$

$$A^n - B^n = (A - B) \cdot (A^{n-1} + A^{n-2}B + ... + B^{n-1}), \qquad n = 2, 3, 4, 5$$

Viète's literal calculus was later perfected by Rene Descartes, who gave the literal calculus its modern form. At the end of the 17th century, a calculus was developed for the analysis of infinitesimals; this is seen in Newton's method of fluxions and infinite series and Leibniz's differential and integral calculus. The 18th century saw the development of a calculus of partial differentials and derivatives, and the 19th century saw the creation of a calculus of logic. Today, nearly every mathematical theory has its own literal calculus; the apparatus of formulas has become an "indispensable language of mathematics," and was introduced by Diophantus and Viète [02], 80.

5.3.3 Viète's determinate equations

Viète's literal calculus allowed for his analysis of determinate equations. His treatise *On Perfecting Equations* establishes what is now known as Viète's Theorem: [02], 87

$$\text{For} \quad x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0, \quad n = 2, 3, 4, 5$$

To have n solutions $x_1, x_2, ...x_n$, the symmetric expressions result:

$$x_1 + x_2 + ... + x_n = -a_1$$

$$x_1x_2 + x_1x_3 + ... + x_{n-1}x_n = a_2$$

....

$$x_1x_2...x_n = (-1)^n a_n$$

Viète's solution of the cubic equation $x^3 + 3ax = 2b$ is presented below because of its implications for the Fundamental Theorem of Algebra:

$$x^3 + 3ax = 2b$$

Let

$$a = t^2 + xt = t(t + x)$$

The elimination of x from these two equations yields a quadratic equation in t^3:

$$(t^3)^2 + 2bt^3 = a^3$$

Thus, cubic equations can be solved by trigonometry. Using the example Viète used: [02], 88-9

$$x^3 - px = q, \quad \text{where} \quad (\frac{q}{2})^2 < (\frac{p}{3})$$

$$\Rightarrow x^3 - 3r^2x = ar^2$$

Since

$$(\frac{ar^2}{2})^2 < (r^2)^3 \quad \Leftrightarrow \quad a < 2r$$

Let

$$a = 2r \sin v$$

Then

$$x^3 - 3r^2x = ar^2 \quad \Leftrightarrow \quad (\frac{x}{r})^3 = 3(\frac{x}{r}) + 2\sin v$$

Let $\frac{x}{r} = -y$:

$$3y - y^3 = 2\sin v$$

Using $3\sin\varphi - 4\sin^4\varphi = \sin 3\varphi$, Viète obtains the equation:

$$3(2\sin\varphi) - (2\sin\varphi)^3 = 2\sin 3\varphi$$

$$\Rightarrow y_1 = 2\sin\frac{v}{3}; \quad y_2 = 2\sin\frac{v + 2\pi}{3};$$

The third root is negative. This showed that "in the 'irreducible' case a cubic equation has three different real roots. Since Viète admitted only positive roots he could not formulate this conclusion explicitly" [02], 89. Thus, Viète showed a part of the conclusion of the Fundamental Theorem of Algebra, though he did not formulate it in the same way as Descartes and Girard did later.

5.4 Algebra in the 17th, 18th centuries

The 16th century was a crucial point in the history of algebra, with the invention of the literal calculus. The use of its own language allowed for investigations of determinate and indeterminate equations. The domain of numbers was extended to complex numbers, though these were not viewed as entirely "legitimate"[02], 90. After these algebraic achievements, a period of relative calm ensued. 17th century mathematicians were primarily engaged in analysis of infinitesimals. However, arithmetization also occurred in the 17th and 18th centuries.

5.4.1 Descartes' arithmetization of algebra and determinate equations

Rene Descartes (1596-1650 CE) was both a philosopher and a mathematician. His *Geometry* was an attempt to reduce geometry to algebra, which created analytic geometry. He transformed Viète's calculus of magnitudes in specifying that "operations on segments should be a faithful replica of the operations on rational numbers" [02], 91. Instead of regarding the product of two segments as an area, as Viète and Greek mathematicians did, Descartes showed that the product is a segment [09], 73. He introduces a unit segment (u) and defined the products of segments a and b as the segment c which was the fourth proportional to the segments u, a, and b; i.e. $u{:}a = b{:}c$. Descartes then made the domain of segments into a replica of the semi-field $\mathbf{R}+$, establishing the isomorphism between the domain of segments and the semi-field $\mathbf{R}+$ [02], 92.

Isaac Newton (1643-1727 CE) later gave a definition of numbers which was omitted by Descartes. Though Greek mathematicians only regarded numbers as collections of units, like natural numbers, Ptolemy and Arab mathematicians identified ratios of numbers, or the rational numbers, and ratios of like quantities, such as real numbers, as numbers. However, European mathematicians did not do so until Newton defined ratios as numbers, following in Descartes example. Newton wrote: [02], 93

> By a 'number' we understand not so much a multitude of units as the abstract ratio of any quantity to another quantity which is considered to be unity. It is threefold: integral, fractional, and

> surd. An integer is measured by unity, a fraction by a submultiple part of unity, while a surd is incommensurable with unity.

Newton also defined negative numbers and operation rules with relative numbers.

Descartes also presented his properties of equations (by "equation," Descartes meant setting a polynomial equal to zero), which lead to the development of the Fundamental Theorem of Algebra: [02], 93-4

> 1. If α is a root of an equation then its left side is algebraically divisible by $x - \alpha$;
> 2. An equation can have as many positive roots as it contains changes of sign from $+$ to $-$; and as many false (i.e. negative) roots as the number of times two $+$ signs or two $-$ signs are found in succession;
> 3. In every equation one can eliminate the second term by a substitution; [as in completing the square in a quadratic equation]
> 4. The number of roots of an equation can [sic] be equal to its degree.

Descartes also formulated assertions of cubic and quartic equations, analyzing their constructibility by ruler and compass (assuming all its roots are real). Descartes discovered the method of undetermined coefficients, which led to his (cautious) formulation of the Fundamental Theorem of Algebra [02], 94.

5.4.2 The Fundamental Theorem of Algebra

Descartes first formulated the Fundamental Theorem of Algebra in the following way: "Every equation can have as many distinct roots (values of the unknown quantity) as the number of dimensions of the unknown quantity in the equation" [02], 94. Girard overcame Descartes' reluctance to include complex roots, and in his *New Discoveries in Algebra* in 1629 wrote that the number of solutions of an algebraic equation is equal to its degree. Other 18th century mathematicians used an equivalent version of the Fundamental Theorem: "Every polynomial

$$f_n(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

with real coefficients can be written as a product of linear and quadratic factors with real coefficients" [02], 95

The first proof of the Fundamental Theorem of Algebra was given by d'Alembert in 1746, but his proof was purely analytic and was not rigorous, even compared to the standards of rigor of the 18th century. Euler (1707-1783 CE) presented his proof of the Fundamental Theorem of Algebra in 1746 as well; Euler's proof differed from d'Alembert's in that Euler looked for a purely algebraic proof. Euler reduced his non-algebraic assumptions to a minimum, using the following two assumptions: [02], 95

> I. Every equation of odd degree with real coefficients has at least one real root.
> II. Every equation of even degree with real coefficients and negative constant term has at least two real roots.

Euler formulated his proof by reducing the solution of an equation of degree $2^k m$, m odd, to an equation of degree $2^{k-1}m_1$, m_1 odd. Euler noted that it is sufficient to consider an equation $P_n(x) = 0$ for $n = 2^k$; if $n \neq 2^k$ then it is possible to find a value of k such that $2^{k-1} < n < 2^k$ and multiply the polynomial $f_n(x)$ by $2^k - n$ factors to get a polynomial of degree 2^k. Euler therefore only proves the theorem for $n = 4$, 8, and 16 and for the general case $n = 2^k$. I will show his cases for $n = 4$ and $n = 2^k$:

Consider the equation: $$x^4 + Bx^2 + Cx + D = 0 \qquad (A)$$

The left side can be written as the product

$$(x^2 + ux + \lambda)(x^2 - ux + \mu)$$

Using Descartes' method of undetermined coefficients gives the equation:

$$u^6 + 2Bu^4 + (B^2 - 4D)u^2 - C^2 = 0$$

According to assumption II, this equation has at least two roots, one of which is the value u. Euler shows that the coefficients λ and μ can be expressed in terms of u and the coefficients in equation (A).

Euler then reaches the same conclusion from general arguments without relying on computations in order to extend his assertion to an equation of degree 2^k. He uses the following (then unproved) theorems of Lagrange and Galois: [02], 96

Theorem 5.4.1. (Fundamental Theorem on Symmetric Functions) Every rational symmetric function of the roots of an equation is a rational function of its coefficients.

Theorem 5.4.2. If a rational function $\varphi(x_1, \ldots, x_n)$ of the roots of an equation takes on k different values under all possible permutations of the roots, then those k values satisfy an equation of degree k whose coefficients are rationally expressible in terms of the coefficients of the initial equation.

Following Descartes, Euler argued that to any equation of degree n we can "attribute" n roots and write:

$$f_n(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = (x + \alpha_1)(x + \alpha_2)\cdots(x + \alpha_n)$$

where α_1, α_2, …, α_n are symbols which operate as if they were ordinary numbers. Thus, the α_i are the negatives of the roots of f_n and thus: [02], 96

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = a_1$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{n-1}\alpha_n = a_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_1\alpha_2\cdots\alpha_n = a_n$$

The Fundamental Theorem of Algebra states that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are real or complex numbers.

Then Euler assumes that equation (A) has roots α, β, γ, and δ. Then, u must be the sum of two of these four roots (so there are six possibilities for u), and so u must satisfy an equation of degree 6. Euler writes that since $\alpha + \beta + \gamma + \delta = 0$, the values of u can be: [02], 97

$$u_1 = \alpha + \beta = p \qquad u_4 = \gamma + \delta = -p$$

$$u_2 = \alpha + \gamma = q \qquad u_5 = \beta + \delta = -q$$

$$u_3 = \alpha + \delta = r \qquad u_6 = \beta + \gamma = -r$$

Then, the equation for u takes the form:

$$(u^2 - p^2)(u^2 - q^2)(u^2 - r^2) = 0$$

This equation is an even degree and has $-p^2q^2r^2$ as its constant term. To verify that $-p^2q^2r^2 < 0$, it is enough to show that pqr is real. Euler shows that:

$$pqr = (\alpha + \beta)(\alpha + \gamma)(\alpha + \delta)$$

remains unchanged under all permutations of the roots, which means that pqr is expressible in terms of the coefficients in equation (A). Thus, u can be chosen to be real.

Euler then sketches the proof for the case $n = 2^k$ [02], 97. He writes the polynomial

$$f_n(x) = x^{2^k} + Bx^{2^k-2} + Cx^{2^k-3} + \cdots \qquad (B)$$

as a product of two factors of degree 2^{k-1} with indeterminate coefficients

$$(x^{2^{k-1}} + ux^{2^{k-1}-1} + \lambda x^{2^{k-1}-2} + \cdots)(x^{2^{k-1}} - ux^{2^{k-1}-1} + \mu x^{2^{k-1}-2} + \cdots)$$

The number of coefficients is $2^k - 1$ and is thus equal to the number of determining relations.

Since u is the sum of 2^{k-1} of the total 2^k roots, the number of possible values of u is $\binom{2^k}{2^{k-1}} = 2N$, where N is odd. Thus, Euler concludes that u satisfies an equation of degree $2N$ with real coefficients. This equation resembles the above example, in this it must be of the form:

$$(u^2 - p_1^2)(u^2 - p_2^2) \cdots (u^2 - p_N^2) = 0$$

This then has a constant term $p_1^2 \cdots p_N^2$, which Euler points out is negative; thus, u can be chosen to be real. The remaining coefficients λ, μ, $\cdots$ can be expressed rationally in terms of u and the coefficients B, C, D, $\cdots$ from equation (B) above.

Lagrange fully accepts Euler's proof, and his *On the Form of Imaginary Roots of Equations* is a rigorous account of Euler's reduction procedure which fills the gaps in Euler's proofs. Lagrange also assumes that one can "attribute" n root symbols to an arbitrary equation of degree n and operate with them under the usual arithmetic rules. Other 18th century mathematicians agreed; the proofs of de Fonxenex (1759) and Laplace (1795) simplified Euler's reduction procedure but "regarded his formulation of the issues as completely legitimate" [02], 98.

The first mathematician who rejected Euler's formulation was C. F. Gauss (1777-1855 CE) [09], 79, 95. His doctoral dissertation, written in 1799, was devoted to the proof of the Fundamental Theorem of Algebra. In it, he wrote: [02], 98

> Since we cannot imagine forms of magnitudes other than real and imaginary magnitudes $a + b\sqrt{-1}$, it is not entirely clear how what is to be proved differs from what is assumed as a fundamental proposition; but granted one could think of other forms of magnitudes, say F, F', F'', ..., even then one could not assume without proof that every equation is satisfied either by a real value of x, or by a value of the form $a + b\sqrt{-1}$, or by a value of the form F, or of the form F', and so on. Therefore the fundamental theorem can have only the following sense: every equation can be satisfied either by a real value of the unknown, or by an imaginary of the form $a+b\sqrt{-1}$, or, possibly, by a value of some as yet unknown form, or by a value not representable in any form. How these magnitudes, of which we can form no representation whatever - these shadows of shadows - are to be added or multiplied, this cannot be stated with the kind of clarity required in mathematics.

Gauss gave a largely algebraic proof without assuming the existence of roots of any form in 1815 [09], 95-9. Kronecker isolated the method of Gauss in pure form with the construction of the splitting field of a polynomial without assuming the existence of the field of complex numbers in 1880-1881. This may be one of the first examples of abstract algebra; however, there is no need to detail this particular proof here, since the first proof of the Fundamental Theorem of Algebra has already been given, and a purely algebraic proof of the Theorem (using Galois Theory) will be provided later.

5.5 Algebra in the 19th century

Though Euler's viewpoint was rejected at the beginning of the 19th century, it was adopted between the 1870s and 1880s, and became the viewpoint that triumphed in algebra over the viewpoint that presupposes the construction of a field of complex numbers which is then followed by a proof of the existence of a root in the field. It is interesting to note that the Fundamental Theorem of Algebra in Euler's proof coincides with the Weierstrass-Frobenius theorem which states that "the field of real numbers and the field of complex numbers are the only linear associative and commutative algebras (without zero divisors) over the field of real numbers" [02], 100. Gauss remained a large part of 19th century mathematics, as his theory of cyclotomic equations became the model for the investigations of Abel, Galois, and other 19th-century algebraists.

5.5.1 Galois Theory

Evariste Galois (1811-1832 CE) solved the problem of algebraic solutions of equations. Galois was killed in a duel on May 30, 1832; the night before the duel, knowing he may die, Galois wrote a letter to his friend Auguste Chevalier setting out his fundamental results [09], 103. These dealt with the general theory of algebraic functions and the theory of equations. It will be helpful to give a brief overview of Galois theory, to aid in the understanding of a pure algebraic proof of the Fundamental Theorem of Algebra, which relies on Galois theory.

Galois theory deals with whether an equation

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \qquad (C)$$

with numerical coefficients a_1, a_2, ..., a_n, is solvable by radicals.

Galois introduces the concepts of field, which he calls "domain of rationality," and group [02], 117. He defines rational as "any quantity which can be expressed as a rational function of the coefficients of the equation and of a certain number of *adjoined* quantities arbitrarily agreed upon" [02], 117. He also introduces the concept of a group of substitutions; "substitutions are the passage from one permutation to another" [02], 117.

For the domain of rationality, or field, Galois uses $\mathbf{Q}(a_1, \ldots, a_n) = \mathbf{Q}_0$, where $\mathbf{Q}$ is the field of rational numbers. He usually adjoins to $\mathbf{Q}_0$ all necessary roots of unity. For example, the equation

$$x^p - a = 0$$

becomes solvable with radicals if the radical $\sqrt[p]{a}$ is adjoined to the domain of rationality.

Galois theory shows that the splitting field $\mathbf{K} = \mathbf{Q}_0(x_1, \ldots, x_n)$ of equation (C) above is obtained by the adjunction of a single element $\theta : \mathbf{K} = \mathbf{Q}_0(\theta)$. The element θ is the primitive element of the field $\mathbf{K}$ [02], 118.

The question of the solvability of equation (C) then reduces to the study of the structure of the field $\mathbf{K}$. In other words, can the field be obtained by successive adjunction of radicals to $\mathbf{Q}$ ($K = Q[\theta_1, \theta_2, ..., \theta_n]$)? Galois reduced the problem to the "study of the structure of a finite group, the group G of automorphisms of the field $\mathbf{K}$ that leave the elements of the groundfield $\mathbf{Q}_0$ fixed" [02], 118. This group G has order g and consists of the permutations

$$\theta_1 \to \theta_1,\ \theta_1 \to \theta_2,\ \theta_1 \to \theta_g$$

This group G is noncommutative, and is known as the Galois group. If H is a subgroup of G, then the elements of $\mathbf{K}$ invariant under the permutations in H form a subfield $\mathbf{L}$. Generally, $\mathbf{L}$ is not normal, or is not determined by a normal equation. To obtain normal equations and normal subfields, Galois realized that he must take only subgroups H such that the cosets H, g_1H, $\ldots$, $g_{S-1}H$ form a group; this is called the factor group G/H. In other words, H must be a normal subgroup of G.

Galois introduces and defines the concepts of a normal subgroup and a factor group. "If H is a normal subgroup of order h and index p and $\mathbf{L}$ is the subfield of $\mathbf{K}$ whose elements are invariant under H, then $\mathbf{K} \supset \mathbf{L} \supset \mathbf{Q}_0$ and, as Galois shows, the degree of $\mathbf{L}$ over $\mathbf{Q}_0$ is p (the order of the factor group G/H and the degree of $\mathbf{K}$ over $\mathbf{L}$ is h (the order of H). This is a direct generalization of Gauss' theory of the cyclotomic equation" [02], 119.

A necessary condition for $\mathbf{K}$ to be obtainable from $\mathbf{Q}_0$ by a finite number of extensions

$$\mathbf{Q}_0 \subset \mathbf{L}_1 \subset \mathbf{L}_2 \subset \cdots \subset \mathbf{L}_S = \mathbf{K}$$

is the existence in G of a nested sequence of normal subgroups

$$G \supset H_1 \supset H_2 \supset \cdots \supset H_S \supset E$$

such that the factor groups G/H_1, H_1/H_2, $\ldots$, H_S have prime orders $p_i, i = 1 \cdots S$ respectively. Further, all of the extensions are radical extensions:

"$\mathbf{L}_1$ is obtained from $\mathbf{Q}_0$ by adjunction of a root of the equation $x^{p_1} - a = 0$, $a \in \mathbf{Q}_0$, $\mathbf{L}_2$ is obtained from $\mathbf{L}_1$ by adjunction of a root of the equation $x^{p_2} - a_1 = 0$, $a_1 \in \mathbf{L}_1$, and so on. Each of these equations has a cyclic Galois group of prime order and is therefore solvable by radicals (we are assuming that the necessary roots of unity have already been adjoined to $\mathbf{Q}_0$)" [02], 119-20.

To obtain a criterion for the solvability of equations by radicals, Galois constructed a "complex chain of interrelated concepts" [02], 120. Galois creates a normal equation from the given equation by constructing a primitive element; to make precise the concept of a group of permutations; to define the Galois group of the equation; and to introduce the concepts of a normal subgroup, of a factor group, and of a solvable group [09], 116.

Chapter 6

Appendix II

6.1 Prerequisites

It is now possible to prove the Fundamental Theorem of Algebra with a purely algebraic proof, using Galois theory. We will use the following Theorems in proving the Fundamental Theorem of Algebra: [07], 396

Theorem 6.1.1. Let F be any field. Then the following statements are equivalent:
(1) F is algebraically closed
(2) $f(x) \in F[x]$ is irreducible if and only if the degree of $f(x) \neq 1$
(3) Every non-constant polynomial in $F[x]$ splits over F
(4) If E is an algebraic extension of F, then $E = F$

Theorem 6.1.2. Let $f(x) \in F[x]$ be an irreducible polynomial over a field F. Then:
(1) If char $F = 0$, then $f(x)$ is separable over F
(2) If char $F = p$, then $f(x)$ is separable over F if and only if $f(x) \neq g(x^p)$ for any polynomial $g(x) \in F[x]$

Theorem 6.1.3. (First Sylow Theorem) Let p be a prime, and let G be a finite group, and suppose $|G| = p^n m$, where $n \geq 1$ and p does not divide m. Then for all k with $1 \leq k \leq n$, G contains at least one subgroup of order p^k (and so contains a Sylow p-subgroup)

Theorem 6.1.4. Fundamental Theorem of Galois Theory Let E be a Galois extension of a field F. For any intermediate field K, $F \subseteq K \subseteq E$, let

$\chi(K) = \mathrm{Gal}(E/K)$. Then:
(1) χ is a one-to-one map from the set of all intermediate fields K to the set of all subgroups of $\mathrm{Gal}(E/F)$
(2) $K = E^{\mathrm{Gal}(E/K)}$
(3) $\chi(E^H) = H$ for all $H \leq \mathrm{Gal}(E/F)$
(4) $[E : K] = |\mathrm{Gal}(E/K)|$
(5) $[K : F] =$ the index of $\mathrm{Gal}(E/K)$ in $\mathrm{Gal}(E/F)$, for which we will use the notation $|\mathrm{Gal}(E/F) : \mathrm{Gal}(E : K)|$
(6) K is a Galois (or normal) extension of F if and only if $\mathrm{Gal}(E/K) \lhd \mathrm{Gal}(E/F)$, in which case $\mathrm{Gal}(K/F) \cong \mathrm{Gal}(E/F)/\mathrm{Gal}(E/K)$
(7) For any two intermediate fields $K_1, /, K_2$ we have $K_1 \subseteq K_2$ if and only if $\chi(K_2) \leq \chi(K_1)$ and thus the lattice of subgroups $H \leq \mathrm{Gal}(E/F)$ is the lattice of intermediate fields $F \subseteq K \subseteq E$, inverted

Corollary 6.1.5. Every polynomial $f(x) \in \mathbf{R}[x]$ of odd degree has a zero in $\mathbf{R}$

Theorem 6.1.6. Let p be a prime, and let G be a finite group, and suppose $|G| = p^n$, where $n \geq 1$. Then G contains for all k with $1 \leq k \leq n$, at least one subgroup of order p^k that is a normal subgroup of a subgroup of order p^{k+1}

Note 6.1.7. The quadratic formula expresses the zeros of any quadratic polynomial in terms of square roots. Thus, by guaranteeing the existence of square roots, the Intermediate Value Theorem guarantees that every polynomial $f(x) \in \mathbf{C}[x]$ of degree 2 has a zero in $\mathbf{C}$.

6.2 Fundamental Theorem of Algebra, Galois-style!

We may now prove the Fundamental Theorem of Algebra [07], 396.

Theorem 6.2.1. The Fundamental Theorem of Algebra The field $\mathbf{C}$ of complex numbers is algebraically closed.

Proof. By Theorem 6.1.1, it suffices to show that any non-constant polynomial $f(x) \in \mathbf{C}[x]$ has a zero in $\mathbf{C}$. Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and let

$$\overline{f(x)} = \overline{a_n x^n} + \overline{a_{n-1}x^{n-1}} + \cdots + \overline{a_1 x} + \overline{a_0} \in \mathbf{C}[x],$$

where $\overline{f(x)}$ is the complex conjugate of $f(x)$. Let $g(x) = f(x)\overline{f(x)}$. Then $g(x) \in \mathbf{R}[x]$ and $f(x)$ has a zero in $\mathbf{C}$ if and only if $g(x)$ has a zero in $\mathbf{C}$. Therefore, it suffices to show that any non-constant polynomial in $\mathbf{R}[x]$ has a zero in $\mathbf{C}$.

Let $g(x)$ be an irreducible polynomial in $\mathbf{R}[x]$.

Note: *An irreducible polynomial is a polynomial that cannot be factored into nontrivial polynomials; in other words, an irreducible polynomial cannot be written as the product of two polynomials, both of a smaller degree.*

Let

$$h(x) = (x^2 + 1)g(x) \in \mathbf{R}[x]$$

and let E be an algebraic extension field of $\mathbf{C}$ that is a splitting field of $h(x)$.

Note: *The extension field E of the field $\mathbf{C}$ is a splitting field for the polynomial $h(x) \in \mathbf{C}[x]$ since $h(x)$ factors completely into linear factors in E and does not factor completely into linear factors over any proper subfield of E containing $\mathbf{C}$.*

Note: *We need to define this $h(x)$ because we want $\mathbf{C}$ to be contained in E. If we use $g(x)$, E is an extension field that splits $g(x)$ but $\mathbf{C}$ is not necessarily contained in E. For example, if $f(x) = ix$ and so $\overline{f(x)} = -ix$ and $g(x) = -x^2$, $g(x)$ splits to $-x$, x, so $\mathbf{R} \subseteq E$ but $\mathbf{C} \nsubseteq E$. Our defined $h(x)$, however, necessarily must include i as it splits, so $\mathbf{C} \subseteq E$.*

Then, E is a Galois extension of $\mathbf{R}$ by Theorem 6.1.2.

Consider the Galois group $\mathrm{Gal}(E/\mathbf{R}) = G$. Let

$$|G| = 2^k m \text{ where } m \text{ is odd.}$$

By the First Sylow Theorem (Theorem 6.1.3), G has a 2-Sylow subgroup P of order 2^k, and $P \lhd G$. By the Galois Correspondence, (Theorem 6.1.4 - Fundamental Theorem of Galois Theory, part 5),

$$[E^P : \mathbf{R}] = |G : P| = m.$$

Thus, E^P is an extension of $\mathbf{R}$ of degree m, which implies that there exists an irreducible polynomial of degree m over $\mathbf{R}$. By Corollary 6.1.5, the only odd m for which this is possible is $m = 1$. Thus

$$|G| = 2^k.$$

Since $\mathbf{R}(i) = \mathbf{C} \subseteq E$, $\mathrm{Gal}(E/\mathbf{C}) = J$ is a subgroup of G and hence, by LaGrange, $|J| = 2^n$ for some $n \leq k$. By Theorem 6.1.6, every group of order p^n has a subgroup of order p^i for every $1 \leq i \leq n$. Thus, if $n > 0$, J has a normal subgroup H of order 2^{n-1}. Then

$$[E^H : \mathbf{C}] = |J : H| = 2.$$

Thus E^H is an extension of $\mathbf{C}$ of degree 2, which implies that there exists an irreducible polynomial of degree 2 over $\mathbf{C}$. In other words, if J is non-trivial, it has a subgroup H of index 2, and we have an irreducible polynomial of degree 2 over $\mathbf{C}$. But, according to Note 6.1.7, this is impossible.

So $n = 0$, $E = \mathbf{C}$, and $\mathbf{C}$ contains the zeros of the polynomial $h(x)$, so $\mathbf{C}$ contains the zeros of our irreducible polynomial $g(x) \in \mathbf{R}[x]$ and $\mathbf{C}$ contains the zeros of our non-constant polynomial $f(x)$.

$\therefore$ by Theorem 6.1.1 the field $\mathbf{C}$ of complex numbers is algebraically closed. $\square$

6.3 Conclusion

Of course, the evolution of algebra does not end with the Fundamental Theorem of Algebra, or with Galois Theory. The development of group theory by Galois was continued by Lagrange, Cauchy, and Camille Jordan, to name just a few. Number theory and commutative algebra continued to be developed beyond the 19th century, and linear and noncommutative algebra was likewise developed in the 19th and 20th centuries. Also, with the introduction of abstract objects such as groups, rings, fields, ideals, matrices, algebras, etc., the influence of algebra has continued to influence all areas of mathematics as well as physics, especially in the area of quantum mechanics [02], 162. However, for the purposes of this paper, it suffices to show that the Arab mathematician's influence extended through European mathematics even to the formulation and proof of the Fundamental Theorem of Algebra.

Bibliography

[01] Al-Daffa, Ali Abdullah. *The Muslim Contribution to Mathematics.* London: Croom Helm Ltd, 1977.

[02] Bashmakova, Isabella, and Galina Smirnova. *The Beginnings and Evolution of Algebra.* transl. Abe Shenitzer, ed. David Cox. Washington, DC: The Mathematical Association of America, 2000.

[03] Berggren, J. Lennart. *Episodes in the Mathematics of Medieval Islam.* New York, NY: Springer-Verlag, 1986.

[04] Berggren, J. Lennart. Mathematics in Medieval Islam, in *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook.* ed. Victor J. Katz. Princeton, NJ: Princeton University Press, 2007.

[05] Boyer, Carl. The Arabic Hegemony, in *A History of Mathematics.* Revised by Uta Merzbach. New York, NY: John Wiley & Sons Inc, 1991.

[06] Mohamed, Mohini. *The Lives and Contributions of Selected Non-Western Mathematicians During the Islamic Medieval Civilization.* Temple University. Ann Arbor, MI: University Microfilms International, 1990.

[07] Papantonopoulou, Aigli. *Algebra, Pure & Applied.* Upper Saddle River, NJ: Prentice Hall, 2002.

[08] Turner, Howard. Forces and Bonds: Faith, Language, and Thought, and Mathematics: Native Tongue of Science, in *Science in Medieval Islam.* Austin, TX: University of Texas Press, 1995.

[09] van der Waerden, B.L. *A History of Algebra: From al-Khwarizmi to Emmy Noether.* New York, NY: Springer-Verlag, 1980.