Pascal's Hexagons

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1 Abstract

This paper deals with some interesting properties of "Pascal's Hexagons", which is a construction modeled after Pascal's triangle. Placing 1s on the upper 3 sides of a hexagon in an integral lattice and applying Pascal's algorithm produces a finite hexagonal array. Closed formulas were developed for the last row and pedagogical and visual generalizations were devised. There is a theorem, entitled Pascal's Hexagon Theorem, that involves inscribing a hexagon in a circle, but that theorem is unrelated to this construction. [3]

2 What is a Pascal's Hexagon?

A Pascal's hexagon is an array of integers in the shape that is constructed in a similar manner to Pascal's triangle.

Construction:

For a hexagon where the first row is length n

$$\begin{array}{ccccccccccccc}
 & & & a_{1,1} & & a_{1,2} & & \cdots & & a_{1,n} & & & \\
 & & a_{2,1} & & a_{2,2} & & \cdots & & a_{2,n} & & a_{2,n+1} & & \\
 & \vdots & \vdots & & & & \vdots & & \vdots & & & \vdots & \\
a_{n,1} & & \cdots & & & & & & & & & \cdots & a_{n,2n-1} \\
 & a_{n+1,1} & & \cdots & & & & & & & \cdots & a_{n+1,2n-2} & \\
 & & \vdots & & & & & & & & \vdots & & \\
 & & & a_{2n-1,1} & & \cdots & & & & a_{2n-1,n} & & &
\end{array}$$

$a_{1,k}$ is given

$a_{j,1} = a_{j+1,1}\ forj \leq n$

$a_{j+1,k} = a_{j,k-1} + a_{j,k}\ forj \leq n$

$a_{j+1,k} = a_{j,k} + a_{j,k+1}\ forj > n$

2.1 Examples

n = 1

1

n = 2

	1		1	
1		2		1
	3		3	

n = 3

		1		1		1		
	1		2		2		1	
1		3		4		3		1
	4		7		7		4	
		11		14		11		

n = 4

			1		1		1		1			
		1		2		2		2		1		
	1		3		4		4		3		1	
1		4		7		8		7		4		1
	5		11		15		15		11		5	
		16		26		30		26		16		
			42		56		56		42			

3 Research Questions

The research was directed at the following two questions:

What is the formula for the largest number in the hexagon?

What is the formula for the sum of the last row in the hexagon?

Generating functions can be used to answer the first question:

Theorem 3.1. *If* $a_0 \, , \cdots \, , a_n$ *and* $b_0 \, , \cdots \, , b_{n+1}$ *are consecutive rows using the Pascal sum, and if* $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ *and* $q(x) = b_0 + b_1 x + \cdots + b_{n+1} x^{n+1}$*. then*

$$q(x) = (1 + x)p(x)$$

Proof. In a Pascal construction, let $a_0, a_1, \cdots, a_n$ define a row then the next row of consecutive entries will be

$$a_0, a_0 + a_1, a_1 + a_2, \cdots, a_{n-1} + a_n, a_n$$

And

$$q(x) = a_0 + (a_0 + a_1)x + (a_1 + a_2)x^2 + \cdots + (a_{n-1} + a_n)x^n + a_n x^{n+1}$$

After a little algebra

$$q(x) = a_0 + a_1x + \cdots + a_nx^n + a_0x + a_1x^2 + \cdots + a_nx^{n+1}$$

$$= p(x) + xp(x)$$
$$= (1+x)p(x) \qquad \text{q.e.d.}$$

The hexagons are constructed similarly to how Pascal triangles are constructed except the first row is length n and two lower triangle regions are ignored.

Theorem 3.2. *The largest entry in a hexagon that starts with 1's is equal to*

$$\sum_{i=\lfloor \frac{n-1}{2} \rfloor}^{n-1+\lfloor \frac{n-1}{2} \rfloor} \binom{2n-2}{i}$$

Proof. Using Theorem 1, the coefficients for the expression $q(x) = (1+x)^{r-1}p(x)$ are equal to the entries in the r^{th} row of the hexagon. With the last row, $r = 2n-1$. If the first row is all 1's, then

$$p(x) = 1 + x + \cdots + x^{n-1}$$

Then the largest number will be in the center entry and be equal to the coefficient for $x^{\lfloor \frac{3n-2}{2} \rfloor}$ term in q(x). The coefficient of thex^r term in $(1+x)^r$ is equal to $\binom{n}{r}$ [1] For the equation $(1+x_0)^{2n-2}(1+x_1+\cdots+x_1^{n-1})$ the coefficient of $x^{\lfloor \frac{3n-2}{2} \rfloor}$ or $x^{n-1+\lfloor \frac{n-1}{2} \rfloor}$ is equal to the sum of the coefficients of terms $x_0^{\lfloor \frac{n}{2} \rfloor}x_1^{n-1}, \cdots,$ $x_0^{n-3+\lfloor \frac{n}{2} \rfloor}x_1^2,\ x_0^{n-2+\lfloor \frac{n}{2} \rfloor}x_1$ therefore the largest entry in a hexagon equals

$$\sum_{i=\lfloor \frac{n-1}{2} \rfloor}^{n-1+\lfloor \frac{n-1}{2} \rfloor} \binom{2n-2}{i}$$

q.e.d.

It is interesting to note that the largest integer increases almost by a factor of four.

$$\sum_{i=0}^{n} \binom{n}{k} = 2^n \ therefore \sum_{i=0}^{2n-2} \binom{2n-2}{i} = 2^{2n-2} = 4^{n-1}$$

$$\sum_{i=\lfloor \frac{n-1}{2} \rfloor}^{n-1+\lfloor \frac{n-1}{2} \rfloor} \binom{2n-2}{i} = 4^{n-1} - \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} \binom{2n-2}{i} - \sum_{i=\lfloor \frac{n-1}{2} \rfloor + n}^{2n-2} \binom{2n-2}{i}$$

It is easy to see why the largest integer increase almost by a factor of four. A simpler formula or a recursion was not found. However, here are few of the things that were tried.

$3 = 4^1 - 1$
$14 = 4^2 - 2$
$56 = 4^3 - 2$
$238 = 4^4 - 18$
$957 = 4^5 - 67$

The largest number took on the pattern 4^{n-1} - ϵ. The search for a formula for the sequence $1, 2, 8, 18, 67, 576, 1394$ using an online sequence database was unsuccessful. [2] However, if n is odd, the largest number $= 4^{n1} - 2 \sum_{i=0}^{\frac{n-1}{2}-1} \binom{2n-2}{i}$

If n is even, the largest number $= 4^{n-1} - 2 \sum_{i=0}^{\frac{n}{2}-2} \binom{2n-2}{i} - \binom{2n-2}{\frac{3n}{2}-1}$

There might be a recursion between the evens or odds, perhaps $l(n+2) = 16l(n) + \epsilon$
In the odd case the sequence for ϵ is $8, 61, 496$ and for the even case the sequence is $14, 130, 1134$. However, a pattern for either sequence was not found.

4 Variation of the hexagon

One variation on the hexagon is to start with a row of integers other than 1. For example:

		1		3		2		
	1		3		5		2	
1		4		8		7		2
	5		12		15		9	
		17		27		24		

One would think that the largest integer would be in the center of the last row, but this is not always the case.

Let $a_0, a_1, \cdots, a_{n-1}$ be the first row then the formula for the first entry in the last row is:

$$\sum_{i=0}^{n-1} a_{n-1-i} \binom{2n-2}{i}$$

and the formula for the second term is:

$$\sum_{i=1}^{n} a_{n-i} \binom{2n-2}{i}$$

and the last row is:

$$\sum_{i=0}^{n-1} a_{n-1-i} \binom{2n-2}{i}, \sum_{i=1}^{n} a_{n-i} \binom{2n-2}{i}, \cdots, \sum_{i=n-1}^{2n-2} a_{2n-2-i} \binom{2n-2}{i}$$

The largest term on the last row can be any of the entries depending on the values in the first row. For $n = 3$, let a_0, a_1, a_2 be the first row and then the last row will be

$$6a_0 + 4a_1 + a_2 \quad 4a_0 + 6a_1 + 4a_2 \quad a_0 + 4a_1 + 6a_2$$

The largest term in the last row is dependent upon which of $5a_0$ or $3a_0+2a_1+3a_2$ or $5a_2$ is the largest.

5 Second Research Question

To find the sum of the last row, one must first find the formula for each term using the same method used to find the largest integer. One can obtain a formula for the first integer in the row which is

$$\sum_{i=0}^{n-1} \binom{2n-2}{i}$$

The formula for the second integer is

$$\sum_{i=1}^{n} \binom{2n-2}{i}$$

There are n integers in the last row and the sum of the last row equals

$$\sum_{i=0}^{n-1} \binom{2n-2}{i} + \sum_{i=1}^{n} \binom{2n-2}{i} + \cdots + \sum_{i=n-1}^{2n-2} \binom{2n-2}{i}$$

This formula can be simplified with some reorganization

$$\begin{aligned}
\sum_{i=0}^{n-1} \binom{2n-2}{i} &= \binom{2n-2}{0} + \binom{2n-2}{1} + \cdots + \binom{2n-2}{n-1} \\
&+ \\
\sum_{i=1}^{n} \binom{2n-2}{i} &= \binom{2n-2}{1} + \cdots + \binom{2n-2}{n-1} + \binom{2n-2}{n} \\
&+ \\
&\vdots \\
&+ \\
\sum_{i=1}^{n} \binom{2n-2}{i} &= \binom{2n-2}{n-1} + \cdots + \binom{2n-2}{2n-2}
\end{aligned}$$

$$= \sum_{j=1}^{n-1}(j+1)\binom{2n-2}{j} + \sum_{j=n}^{2n-2}(2n-1-j)\binom{2n-2}{j}$$

Theorem 5.1. $\binom{2n-2}{j} = \binom{2n-2}{2n-2-j}$ [1]

Using Theorem 5.1 and $2n-1-(2n-2-j) = j+1$ one finds

$$\sum_{j=0}^{n-1}(j+1)\binom{2n-2}{j} + \sum_{j=n}^{2n-2}(2n-1-j)\binom{2n-2}{j} = 2\left(\sum_{j=0}^{n-1}(j+1)\binom{2n-2}{j}\right) - n\binom{2n-2}{n-1}$$

Subtract out the $j = n-1$ term is necessary to avoid be counted twice.

Let $m = n-1$

$$2\left(\sum_{j=0}^{m}(j+1)\binom{2m}{j}\right) - (m+1)\binom{2m}{m}$$
$$= \sum_{j=0}^{m} 2j\binom{2m}{j} + \sum_{j=0}^{m} 2\binom{2m}{j} - (m+1)\binom{2m}{m}$$
$$= \sum_{j=0}^{m} 2j\binom{2m}{j} + \sum_{j=0}^{2m}\binom{2m}{j} + \binom{2m}{m} - (m+1)\binom{2m}{m}$$

Theorem 5.2 $\sum_{j=0}^{m} 2\binom{2m}{j} = \sum_{j=0}^{2m}\binom{2m}{j} + \binom{2m}{m}$

Proof $\binom{2m}{m-i} = \binom{2m}{m+i}$

it follows $\sum_{j=0}^{m}\binom{2m}{j} = \sum_{j=m}^{2m}\binom{2m}{j}$ q.e.d

Lemma 5.1 $\sum_{i=0}^{n}\binom{n}{k} = 2^n$ [1]

Theorem 5.3 $\sum_{j=0}^{m} 2j\binom{2m}{j} = m2^{2m}$

Proof $\sum_{j=0}^{m} 2j\binom{2m}{j} = \sum_{j=0}^{m} 2(2m)\binom{2m-1}{j-1}$

Let $k = j-1$

$$= \sum_{k=1}^{m-1} 2(2m)\binom{2m-1}{k}$$
$$= \sum_{k=0}^{2m-1}(2m)\binom{2m-1}{k}$$
$$= 2m2^{2m-1}$$

$= m2^{2m}$ q.e.d

Using Theorem 5.2 and Theorem 5.3 we find the sum of the last row to be

$$= \sum_{j=0}^{m} 2j\binom{2m}{j} + \sum_{j=0}^{2m}\binom{2m}{j} - (m)\binom{2m}{m}$$
$$= m2^{2m} + 2^{2m} - m\tbinom{2m}{m}$$
$$= (m+1)2^{2m} - m\tbinom{2m}{m}$$

or

$$n4^{n-1} - (n-1)\binom{2n-2}{n-1}$$

This is the formula for the sum of the last row of the n^{th} hexagon, where the first row is all 1's.

6 Another triangle

A triangle can be formed out of the last rows of the hexagons

```
                              1
                         3         3
                    11        14        11
               42        56        56        42
          163       218       238       218       163
     638       847       957       957       847       638
2510      3301      3784      3939      3784      3301      2510
```

There was no simple way found to generate this triangle.
Let $T(r, e)$ = the entry in the triangle in the r^{th} row and in the e^{th} position
For example, $T(3, 2) = 14$
To find the first integer in a row

$$T(r+1, 1) = 4T(r, 1) - \frac{2(r-1)!}{(r-1)!r!}$$

To find the rest

$$T(r, e+1) = T(r, e) - \binom{2(r-1)}{e-1} + \binom{2(r-1)}{e-1+r}$$

The reader can prove this formula by examining the summation formulas. Other simpler methods were tired but failed. One such method was:
Finding a pattern for the difference between the first and second entry in each row resulting in the following sequence $3, 14, 55, 299, 791, 3002$ but a pattern was

not found.
Another pattern tired was

$$T(r,2) = T(r,1) + T(r-1,1) + T(r-2,1) + \epsilon$$

the resulting sequence for ϵ was $0, 2, 4, 350, 214$ but I couldn't find a pattern or formula.

7 Other properties

Here are some interesting properties, which can be left to the reader to prove. Modifying the notation earlier, let $a_{n,r,e}$ be the e^{th} entry in r^{th} row of the n^{th} hexagon. Let $\Sigma(n,r)$ be the sum of the r^{th} row in the n^{th} hexagon.
$a_{n,n,n} = 2^{n-1}$
$a_{n+1,n+2,n} = 2a_{n,n+1,n-1} - 1$
$\Sigma(n,r) = n2^{r-1}\ if\ r \geq n$
$\Sigma(n+1,r) = \Sigma(n,r) + 2^{r-1}\ if \geq n$
$\Sigma(n,n+1) = n2^n - 2$
$\Sigma(n,n+2) = n2^{n+1} - 2 - 2(n+2)$
$\Sigma(n,n+3) = n2^{n+2} - 2 - 2(n+3) - (n+2)(n+3) + 2$
$\Sigma(n,n+4) = n2^n + 3 - 2 - 2(n+4) + 2 - 2(\frac{(n+2)(n+3)(n+4)}{6} + n + 4)$
Let $l(n,r) =$ the largest entry in the r^{th} row of the n^{th} hexagon.
$l(n,r+1) = 2l(n,r)\ if\ r \leq n$
$l(n,n+2) = 2l(n,n+1)$
$l(n,n+i) = 2l(n,n+i-1)\ if i\ is\ even$
if you write out the largest number in every row
2^0
2^1
$\vdots$
2^{n-1}
$2^n - 1$
$2^{n+1} - 2$
$2^{n+2} - 4 - n$
$2^{n+3} - 8 - 2n$
$2^{n+4} - 16 - 4n - \frac{n(n+3)}{2}$
$2^{n+5} - 32 - 8n - n(n+3)$
$2^{n+6} - 64 - 16n - 2n(n+3) - \frac{n(n+4)(n+5)}{6}$
$2^{n+7} - 128 - 32n - 4n(n+3) - \frac{n(n+4)(n+5)}{3}$
$2^{n+8} - 256 - 64n - 8n(n+3) - \frac{2n(n+4)(n+5)}{3} - \frac{n(n+5)(n+6)(n+7)}{24}$

After $n+1$ rows in every other row the largest number doubles, but the next row it doubles and then a term is subtracted. The new row will have another term with n, and if the last term started $n(n+i)\cdots$ the terms so far are

n, $\frac{n(n+3)}{2!}$, $\frac{n(n+4)(n+5)}{3!}$, $\frac{n(n+5)(n+6)(n+7)}{4!}$ the next term should be $\frac{n(n+6)(n+7)(n+8)(n+9)}{5!}$. If you look at a hexagon it is obvious why it doubles every other row.

8 Future work

There are several open questions that need to answer. One is proving the pattern mentioned above. Another is to find a recursion with the largest number in the hexagon. Another is to deal with the case where the first row isn't all 1's. Also to find a rule to determine which integer will be the largest in the last row, given the first row.

9 Acknowledgments

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