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1 2D Geometries

1.1 What is geometry?

- Game Landscapes - this is not geometry!
- Euclidean Geometry
- Symmetry

1.2 Classical Approach

Euclidean vs. Non-Euclidean

	Spherical/Elliptic	Euclidean	Hyperbolic
Model Pictures			
Model/Example	S^2, $\mathbb{C}$	$\mathbb{R}^2$	H^2
Angle Sum	$> \pi$	$= \pi$	$< \pi$
Perimeter, C_r	$< 2\pi r$	$= 2\pi r$	$> 2\pi r$
Curvature, K	> 0	$= 0$	< 0

Table 1: Comparison of 2D Geometries

These 3 geometries can be used to model various compact 2D spaces as follows: spherical geometry models only spherical geometry (in the obvious way;) Euclidean geometry models the 1-torus by identifying edges; hyperbolic geometry models n-torus for $n > 1$ in a similar way.

2 3D Geometries

2.1 Classification Theorem for 3D Manifolds

Thurston - Geometrization Conjecture (1982)
Perelman - Proof using Ricci Flows (2004?)

Every three dimensional manifold can be locally modeled with one of the following geometries:

Isotropic	S^3	$\mathbb{R}^3$	H^3
Anisotropic	$S^2 \times \mathbb{R}$		$H^2 \times \mathbb{R}$
		Nil	$\widetilde{SL_2}(\mathbb{R})$
		Sol	

Table 2: List of 3D Geometries

Symmetry Group	$E(3)$	$O(3, \mathbb{R})$	$SO(3, \mathbb{R})$
	$S^2 \times \mathbb{R}$		$H^2 \times \mathbb{R}$
		Nil	$\widetilde{SL_2}(\mathbb{R})$
		Sol	

Table 3: List of 3D Geometries

3 Nil Geometry

The **Nil** geometry is also known as Heisenberg geometry. It is the geometry of the Heisenberg group. This geometry can be thought of as both a Riemannian manifold and sub-Riemannian manifold.

3.1 Manifolds

A manifold, M, is a structure that locally resembles some Euclidean space. At every point p on the manifold there is a tangent space, $\mathrm{T}_p(M)$, describing the possible directions of movement. A Riemannian manifold is one in which $\mathrm{T}_p(M) = S^n$ for some sphere n — that is, every possible direction is allowed; a sub-Riemannian manifold is one where some directions are disallowed.

3.2 Nil as a sub-Riemannian Manifold

$M = (x, y, t)$
Nil Geometry as a sub-Riemannian manifold is completely nonintegrable[1].

$$\mathrm{T}_p(M) = H_p = \text{span}\,\{(1, 0, 2y), (0, 1, -2x)\}$$

On the t axis, with points vertically separated, the space behaves as if it has infinite positive curvature.

Off the t axis, with points without purely vertical separations, the space behaves as if it has infinite negative curvature.

[1]If you're interested in what this actually means, see the appendix courtesy of Dr. Tyson.

Points differing only in the t dimension behave like antipodal points.
Geodesics minimize the area of the figure projected down onto (x, y).
The space is complete and connected.

3.3 Relationship between the Riemannian and sub-Riemannian Picture

The Riemannian picture of **Nil** geometry has $\mathbb{R}^3$ as the underlying space with a metric setting the length of the vectors $(1, 0, 2y), (0, 1, -2x), (0, 0, 1)$ to 1. All directions of movement are allowed.

Tyson: In fact, one can consider a whole family of Riemannian metrics on Nil by fixing ϵ positive and declaring $(1, 0, 2y)$, $(0, 1, -2x)$ and $(0,0,\epsilon)$ to be unit vectors. The case $\epsilon = 1$ is the traditional Riemannian metric on Nil. Philosophically, as $\epsilon \to 0$ this sequence of Riemannian manifolds converges to the sub-Riemannian geometry of the Heisenberg group. This vague statement can be made precise using the language of "Gromov-Hausdorff convergence" (don't worry about what this means.) Intuitively, what we are doing is penalizing motion in the t-direction, making it more and more expensive to travel in this direction. In the limit, it becomes impossible to travel (infinitesimally) in this direction and we are forced to take the "long way around" along the sub-Riemannian geodesics.

It turns out that these ϵ Riemannian metrics are related to each other by scaling. Nil equipped with one of these metrics is *not* a self-similar space: it "looks different depending on how far away you're standing." In fact, as one moves farther and farther away, the effect is to decrease the value of ϵ. So another perspective is that the sub-Riemannian geometry of Nil is what you see when you look at the Riemannian Nil from "infinitely far away." It would be very interesting to have some visual clarification / evidence for this statement.

4 Project Ideas

There are currently implementations of all isotropic three dimensional geometries in OpenGL; further, there are semi-complete implementations of $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$.

- Implement **Nil** in OpenGL!
- Driving on geodesics with a steering wheel

- Moving a pointer, with its tangent space attached, in **Nil** and seeing the tangent space deform
- Above, with movement of pointer constrained to directions in tangent space

A A Discussion of Complete Nonintegrablity

The term "completely nonintegrable" refers to a certain geometric/algebraic property of the tangent spaces. Practically speaking, it's the property which allows us to conclude that any two points can be connected with curves staying tangent to the "horiz"[2] planes H_p. The precise notion - in case you're interested - goes as follows:

Identify the vectors $(1, 0, 2y)$ and $(0, 1, -2x)$ with first-order differential operators:

$$X = (d/dx) + 2y(d/dt) and Y = (d/dy) - 2x(d/dt).$$

An important notion in differential geometry is the "commutator" or "Lie bracket" of two such operators: $[X, Y] = XY - YX$. Let's calculate:

(note use of the product rule in the last step). Similarly,

Then

What's important is that this is not zero - X and Y do not commute. Geometrically, the meaning of this is that if we move (infinitesimally) first in the X direction, then in the Y direction, then in the X^{-1} direction, then in the Y^{-1} direction, we end up with a motion in the "disallowed" t-direction. (Think of the examples of parallel parking, motion inside a parking garage, etc.)

By way of contrast, consider a different situation: operators A=(d/dx) and B=(d/dy). These commute of course - equality of mixed partial derivatives. In this case the allowed directions are always *horizontal* in the usual sense of the word, i.e., the xy-plane at every point. Obviously, it's not possible to join two points with different t-coordinates with a path which is always tangent to this family of planes. This is the "integrable" case.

In 3D, with a family of 2D planes, the only possibilities are integrable and completely nonintegrable. (There's only one missing direction.) In higher dimensions, one could imagine intermediate cases, when one picks up some (but not all) of the missing directions via commutators.

[2]Describe.