illiSol: A Modern Orrery

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Introduction

The goal of the illiSol Project is to create a modern and reasonably accurate visualization of the Solar System. The main idea is to present the concept of scale in the universe - initial settings of the visualization provide an accurately scaled model, in which, of course, it is nearly impossible to find anything except Sol itself. Because of this, the visualization provides tools to alter the scaling of both time and space to the viewer's whim, making it a robust and reasonably accurate tool for celestial visualization within the Solar System.

Perhaps the most unique feature of illiSol is its ability to run in immersive virtual environments running on the Syzygy environment - namely, the CUBE and the CAVE at the University of Illinois at Urbana-Champaign. This places the viewer *within* the Solar System with the ability the fly about and view the major moons of Jupiter, or observe the most elliptic orbit in the system of Mercury, enabling interactive and immersive visualization.

I feel that the naming convention deserves at least a token explanation. In Spring 2008, I was an undergraduate in a "Hypergraphics" course under the direction of Dr. George K. Francis, who has been the mentor for the project. The course investigated visualizations of mathematical phenomena using computing technology - the students wrote programs, almost uniformly using the Syzygy environment, to aid visualization in their field of interest. My project, which has grown up from a semester project into its own small application, was, of course illiSol.

The prepending of "illi" to the front of application names arises, as you may have noticed, from their origin at the University of Illinois. Over the years, Dr. Francis' program has been renamed occasionally to reflect its nature. The current program, now coming to a close as of this writing at the end of Summer 2008, is *illiMath '08.* I have followed suit by combining the application's origins into one simple name.

This paper serves as a brief introduction to the application itself, but is also intended to describe the real science behind the Solar System - starting with birth of modern celestial mechanics and culminating with the very set of equations, provided by Jet Propulsion Labs, that runs this visualization. My hope is that seeing this visualization work from the ground up, in concert with this paper, will provide a simple but meaningful introduction to the broad field of celestial mechanics, a field which I, myself, have been interested in since childhood.

A Brief Overview

This section details some of the basic features of the illiSol Project. I will not describe here how to use these features - that is for a different time and a different place. Instead, I will cover the theory behind the features in this overview.

As touched upon in the introduction, one of the defining features of the illiSol Project is that it allows the viewer to alter the scaling of time and space, for the most part, at whim. The desktop version, for instance, provides multiple methods for altering the environment - the viewer can speed up and slow down time (which begins on the launch date of the Voyager II), jump back exactly one Earth-year in time, and scale the radii of the planets and the Sun in both uniform and non-uniform methods. Due to the limited-input nature of the virtual environment controls, the scalability controls are somewhat stripped down to the basics - scaling of the radii can only be done in a non-uniform fashion, but the viewer can still speed up and slow down time as he or she pleases, as well as go back in time one Earth-year. In addition, the ability to follow the reconstructed path of the Voyager II has been added to provide viewers with a first-person simulation of what the Voyager saw it its travels through the Solar System.

All of this is founded upon a series of equations provided by the Jet Propulsion Lab[1], which enables us to approximate the position of every major planet in the Solar System given the date (to be precise, the number of days past the Julian Ephemeris Date, but the conversion from the Gregorian Calendar is straightforward). I will cover these equations in detail in the **Mathematics** section. However, none of this would be available to us without the ingenuity and work of Brahe, Kepler, and, of course, Newton. The rich history that is the foundation of celestial mechanics will be covered first in the **History** section, which is largely founded on the writings of Otto Toeplitz[8]. My goal is to "connect the dots" between the two sections, but this is difficult given both the limited breadth of my knowledge on the subject and a lack of space (this was, after all, not meant to be a textbook).

To accomplish this I will use illiSol as a sort of case study - the source code, available on my webpage[2] is an example of how these equations "work" in a computational context (as we will see, the JPL equations involve integrating Kepler's Equation using an iterative method). If you want to get to the real gears of the visualization, you may just want to skip to the **Mathematics** section. Otherwise, we will start with the birth of modern celestial mechanics.

[1] You can find them at `http://ssd.jpl.nasa.gov/?planet_pos`

[2] `http://new.math.uiuc.edu/illimath/wldavis29/illisol/`

Historical Mathematics - Newton's Law of Gravitation

In the 17th century, mankind's understanding of celestial mechanics began to change and develop. It had not been so long (about. a century) since Copernicus had proposed a heliocentric model for the solar system[5], which differed greatly from Ptolemy's universally accepted geocentric universe. In Ptolemy's model, the stars were fixed on a celestial sphere that rotated around the earth - the planets (that were visible to the naked eye, of course) and the Sun also resided on concentric spheres but instead of being fixed, they moved in a circular path on their respective spheres [3].

With observational data from Tycho Brahe[2], however, Kepler was able to qualify Copernicus' conclusions that the Earth was, indeed, part of a heliocentric system, though Brahe himself believed that the solar system was geocentric[7]. From Brahe's data, Kepler derived his three laws [8]:

1. *The radius vector extending from the sun to a planet sweeps out equal areas in equal times.*
2. *The planets describe ellipses around the sun, which stands in one focus of the elliptical orbit.*
3. *The squares of the periods of revolution of the planets are to each other as the cubes of their major axes.*

While Kepler's Laws were vital in the development of celestial mechanics, the fact still remained that they were observational, not proven. In fact, in the case of the Third Law, Kepler was searching not for a definitive scientific law, but instead a perfect ratio that governed their existence[8]. What is most important to note is what Isaac Newton did next.

Isaac Newton's method in developing his Universal Law of Gravitation would ultimately become the model for theoretical physical research, as Toeplitz puts it[3]. Instead of making assumptions to begin with, he took facts obtained by Kepler and applied his mastery of calculus and geometry to develop his own Law of Gravitation, and subsequently showed that Kepler's Laws were not only compatible with the Law of Gravitation, but were proven by it.

To begin as Newton did, we will take Kepler's first two laws, which were observationally derived, as fact. With the goal of finding a law to govern the motion of the planets, we need to find their acceleration, which Newton thought to be a vector quantity, with a direction and a magnitude. We will first find the direction of acceleration, followed by the magnitude.

Direction of Planetary Acceleration

Putting Kepler's first law in terms of differential calculus, we can obtain the following:

1. *The radius vector extending from the sun to a planet sweeps out equal areas in equal times, so that if A is the aforementioned area expressed in polar coordinates,*

[3]The rest of this section is largely referenced from his book [8]

$$\frac{dA}{dt} = k$$

where t is time and k is constant, and thus,

$$\frac{d^2A}{dt^2} = 0$$

However, the area swept out by the radius in time dt, $\frac{dA}{dt}$, can be approximated.

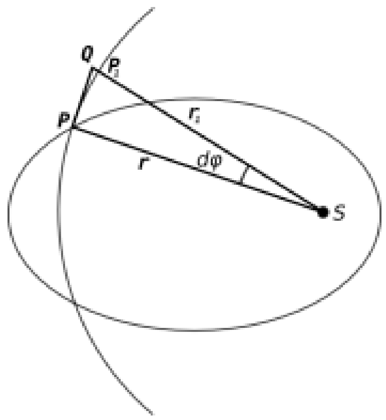

Figure A - Recreated from Toeplitz's *The Calculus*

Figure A shows an "enlarged" view of the area dA. The arc shown intersecting the ellipse at P is a portion of the circle of radius r centered at the focus, S. In the case where $d\varphi$ becomes infinitesimally small, the difference in area between the right triangle SPQ (PQ is tangent to the circle and thus perpendicular to r) and the sector of the ellipse from r_1 to r, dA, is negligible. This triangle, however, can further be approximated by the sector of the circle from P_1 to P, and because we are using polar coordinates[4],

$$dA = \mathit{Area\ of\ SPQ} = \frac{1}{2}r^2 d\varphi$$

Now, because we are concerned with how the system changes with respect to time,

$$\frac{dA}{dt} = k = \frac{1}{2}r^2\frac{d\varphi}{dt}$$

If we differentiate once more with respect to t, we find the following:

$$\frac{d^2A}{dt^2} = \left(\frac{1}{2}r^2\frac{d\varphi}{dt}\right)\frac{d}{dt}$$

We will switch notation now to clean things up a bit, since we will always be differentiating with respect to time. We can also simplify things a bit by using Kepler's First Law.

$$\frac{d^2A}{dt^2} = r\dot{r}\dot{\varphi} + \frac{1}{2}r^2\ddot{\varphi} = 0$$
$$\frac{d^2A}{dt^2} = 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0$$

[4]Note that though we are using an ellipse as our example, this is not necessary yet. We could use a circle to prove our point, in which case no approximation would be necessary (but then we would be in uniform circular motion, which isn't as interesting).

For a moment we will digress from the path to review polar coordinates:

$$x = r\cos\varphi \qquad y = r\sin\varphi$$
$$r = \sqrt{x^2+y^2} \qquad \tan\varphi = \tfrac{y}{x}$$

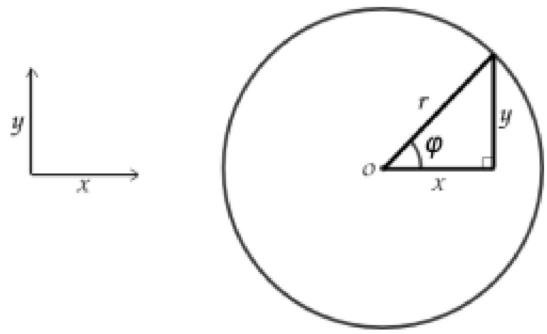

Figure B - Polar Coordinates

If we describe x and y as functions of time, (i.e. $x = x(t)$ and $y = y(t)$), and use these two functions to describe the position of a particle in the xy-plane, we can easily determine the velocity and acceleration in each direction.

$$\dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \qquad \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$$
$$\ddot{x} = \ddot{r}\cos\varphi - \dot{r}\dot{\varphi}\sin\varphi - \dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi$$
$$\ddot{y} = \ddot{r}\sin\varphi + \dot{r}\dot{\varphi}\cos\varphi + \dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\cos\varphi - r\dot{\varphi}^2\sin\varphi$$

which can be simplified as

$$\ddot{x} = (\ddot{r} - r\dot{\varphi}^2)\cos\varphi - (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\sin\varphi$$
$$\ddot{y} = (\ddot{r} - r\dot{\varphi}^2)\sin\varphi + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\cos\varphi$$

In order to determine the direction of the acceleration, ψ, we can see that this a simple trigonometric relationship:

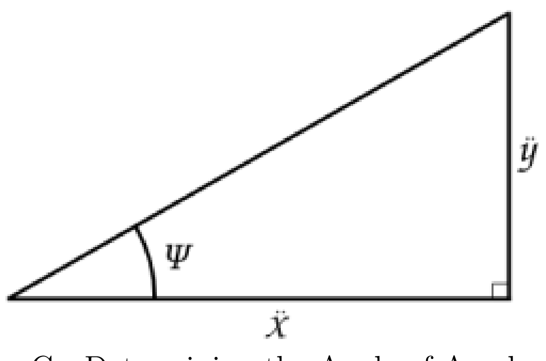

Figure C - Determining the Angle of Acceleration

$$\tan\psi = \frac{\ddot{y}}{\ddot{x}}$$
$$\tan\psi = \frac{\ddot{y}}{\ddot{x}} = \frac{(\ddot{r}-r\dot{\varphi}^2)\sin\varphi + (2\dot{r}\dot{\varphi}+r\ddot{\varphi})\cos\varphi}{(\ddot{r}-r\dot{\varphi}^2)\cos\varphi - (2\dot{r}\dot{\varphi}+r\ddot{\varphi})\sin\varphi}$$

We can use the equation above in the context of our elliptical orbit because we stated r and φ as functions of time. We can therefore use this generalization of acceleration, where ψ is the direction of acceleration, whenever we use polar coordinates, as long as we have r and ψ in terms of t. Now, because

$$\frac{d^2A}{dt^2} = 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0$$

as shown earlier, we see that the last term of both the numerator and denominator drop out, leaving

$$\tan\psi = \frac{\ddot{y}}{\ddot{x}} = \frac{(\ddot{r} - r\dot{\varphi}^2)\sin\varphi}{(\ddot{r} - r\dot{\varphi}^2)\cos\varphi}$$

which simply reduces to

$$\tan\psi = \frac{\sin\varphi}{\cos\varphi} = \tan\varphi$$
$$\psi = \varphi, \varphi + \pi$$

Therefore, since ψ, the direction of acceleration measured from an initial angle, is equal to φ, the angle the radius has swept through starting at the same initial angle, the direction of acceleration in the case of our ellipse is directed inward toward the central body S. However, because ψ could also equal $\varphi + \pi$, the direction of acceleration of the body in motion could also be directly *away* from the central body. An acceleration directly away from the central body, however, can be ruled out by intuition. This would cause the planets to always be moving away from the central body (of course), which is undermined by observation even in that day and age.

Toeplitz shows in his book that the reverse can be proven - that is, that if the direction of acceleration is inward toward the central body, then the radius vector of the motion must sweep out equal areas in equal time. If S, in our case, lies at the origin, and we start with $\tan\psi = \tan\varphi$, using the equation we derived earlier,

$$\tan\psi = \frac{(\ddot{r} - r\dot{\varphi}^2)\sin\varphi + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\cos\varphi}{(\ddot{r} - r\dot{\varphi}^2)\cos\varphi - (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\sin\varphi} = \tan\varphi$$

we can reduce this very easily:

$$\tan\varphi\{(\ddot{r} - r\dot{\varphi}^2)\cos\varphi - (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\sin\varphi\} = (\ddot{r} - r\dot{\varphi}^2)\sin\varphi + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\cos\varphi$$
$$(\ddot{r} - r\dot{\varphi}^2)\sin\varphi - (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\frac{\sin^2\varphi}{\cos\varphi} = (\ddot{r} - r\dot{\varphi}^2)\sin\varphi + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\cos\varphi$$

Rearranging, we get

$$(\ddot{r} - r\dot{\varphi}^2)\sin\varphi - (\ddot{r} - r\dot{\varphi}^2)\sin\varphi = 2(2\dot{r}\dot{\varphi} + r\ddot{\varphi})(\cos\varphi + \frac{\sin^2\varphi}{\cos\varphi})$$

The left side reduces to zero, and we can pull a cosine out of the denominator on the right side to get the identity $\cos^2\varphi + \sin^2\varphi = 1$:

$$0 = (2\dot{r}\dot{\varphi} + r\ddot{\varphi})(\cos^2\varphi + \sin^2\varphi)\frac{1}{\cos\varphi} = (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\frac{1}{\cos\varphi}$$

Multiplying through by a cosine gives us the expression we found earlier for the second derivative of area swept out by the radius with respect to time, but also tells us that this is equal to zero':

$$\frac{d^2A}{dt^2} = 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0$$

implying that

$$\frac{dA}{dt} = k$$

where k is constant, thus reconciling our result. Newton used this fact to generalize Kepler's first law with the following Area Theorem:

> *A motion in the plane has its acceleration directed toward a fixed central body if and only if the radius vector from the central body to the object in motion covers equal areas in equal times.*[5]

Magnitude of Planetary Acceleration

Note that so far, we have only used Kepler's first law to find the direction of acceleration of the central body (remember that we, like Newton, are looking for a physical law to govern the motion of the planets). Now, we will find the magnitude of the acceleration using the geometry of the ellipse. By expressing the ellipse in polar form:

$$r = \frac{p}{1+\epsilon\cos\varphi}$$

where r is the distance from the planet to the central body, p is the semi-latus rectum of the ellipse, ϵ is the eccentricity of the ellipse, and φ is the angle that the radius vector makes with segment SA in Figure D below.

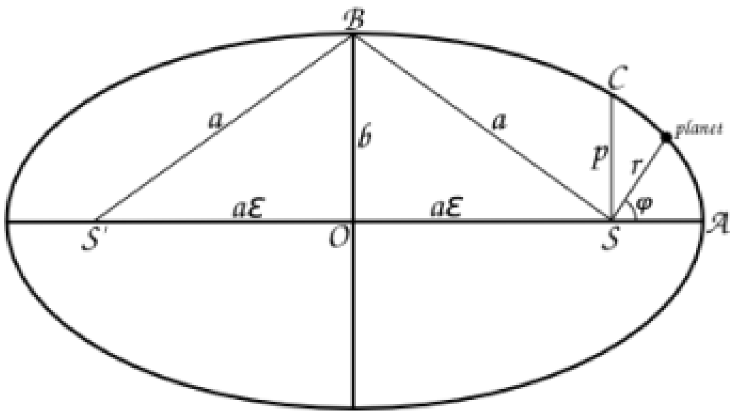

Figure D - The Geometry of an Ellipse

The a and b segments are the lengths of the semimajor and semiminor axes, respectively, shown as such here to illustrate the "string construction" of the ellipse - that is, if the point describing our planet in question were point P', then, because of the nature of ellipses, for any point P' on the ellipse, $SP' + S'P' = 2a$.

We previously obtained the expressions for the acceleration in both the x and y directions to find the angle of acceleration. We can now reuse those same expressions to find the magnitude of the acceleration, J:

$$J^2 = \ddot{x}^2 + \ddot{y}^2 =$$
$$\{(\ddot{r} - r\dot{\varphi}^2)\cos\varphi - (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\sin\varphi\}^2 + \{(\ddot{r} - r\dot{\varphi}^2)\sin\varphi + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\cos\varphi\}^2$$

[5]Paraphrased from Toeplitz

which, with some algebra and simple trigonometry, simplifies to

$$J^2 = \ddot{x}^2 + \ddot{y}^2 = (\ddot{r} - r\dot{\varphi}^2)^2 + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})^2$$

However, since we've proven Kepler's first law, we know that $2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0$, so we find that

$$J^2 = (\ddot{r} - r\dot{\varphi}^2)^2$$

which, by itself, is not useful for us, as we want an expression for the magnitude of acceleration in terms of r, not φ. By doing some algebra and using the fact that $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\varphi} = k$, we can remove φ from the equation:

$$\ddot{r} - r\dot{\varphi}^2 = \ddot{r} - \frac{r^4\dot{\varphi}^2}{r^3} = \ddot{r} - \frac{(r^2\dot{\varphi})^2}{r^3} = \ddot{r} - \frac{4(\frac{1}{2}r^2\dot{\varphi})^2}{r^3} = \ddot{r} - \frac{4k^2}{r^3}$$

We still, however, need to find an expression for $\ddot{r}$ - we can use the polar equation of the ellipse for this (note that eccentricity, ϵ , and the semi-latus rectum, p, are constant):

$$r = \frac{p}{1+\epsilon\cos\varphi}$$
$$r(1 + \epsilon\cos\varphi) = p$$
$$\frac{1+\epsilon\cos\varphi}{p} = \frac{1}{r}$$
$$\frac{\epsilon\cos\varphi}{p} = \frac{1}{r} - \frac{1}{p}$$

Now, differentiate with respect to time (r and φ are both functions of t):

$$-\frac{\epsilon\dot{\varphi}\sin\varphi}{p} = -\frac{\dot{r}}{r^2}$$
$$\dot{r} = \frac{(r^2\dot{\varphi})\epsilon\sin\varphi}{p}$$

Note that Kepler's first law is easily applied here:

$$\dot{r} = \frac{2(\frac{1}{2}r^2\dot{\varphi})\epsilon\sin\varphi}{p}$$
$$\dot{r} = \frac{2k\epsilon\sin\varphi}{p}$$

Differentiate again with respect to time to get $\ddot{r}$:

$$\ddot{r} = \frac{2k\epsilon\dot{\varphi}\cos\varphi}{p}$$

We can obtain $\dot{\varphi}$ from Kepler's first law

$$\frac{1}{2}r^2\dot{\varphi} = k$$
$$\dot{\varphi} = \frac{2k}{r^2}$$

so that

$$\ddot{r} = \frac{2k\epsilon\cos\varphi}{p} \cdot \frac{2k}{r^2}$$
$$\ddot{r} = \frac{4k^2\epsilon\cos\varphi}{pr^2}$$

Because of the periodic nature of orbits, we can see that if we substitute this expression for $\ddot{r}$ back into the original equation for J, we will be able to fully eliminate φ:

$$\ddot{r} - \frac{4k^2}{r^3} = \frac{4k^2\epsilon\cos\varphi}{pr^2} - \frac{4k^2}{r^3} = \frac{4k^2}{r^2}\left(\frac{\epsilon\cos\varphi}{p} - \frac{1}{r}\right)$$

As we saw before,

$$\frac{\epsilon\cos\varphi}{p} = \frac{1}{r} - \frac{1}{p}$$
$$\frac{\epsilon\cos\varphi}{p} - \frac{1}{r} = -\frac{1}{p}$$

so

$$\ddot{r} - \frac{4k^2}{r^3} = \frac{4k^2}{r^2}\left(-\frac{1}{p}\right) = -\frac{4k^2}{r^2 p}$$

If $p > 0$, then,

$$J = \sqrt{(-\tfrac{4k^2}{r^2 p})^2} = \frac{4k^2}{r^2 p}$$

This equation now begins to look familiar, although we often now describe planetary motion as an attractive force between two celestial bodies:

$$F = \frac{Gm_1 m_2}{r^2}$$

Newton's Law

If Kepler's first two laws hold true for a given planet, the acceleration of that planet is always directed towards the central body with a magnitude that is inversely proportional to the square of the planet's distance from said central body.[6]

One can conversely show, as Toeplitz does, that Kepler's laws can be derived from Newton's Universal Law of Gravitation, though I will not do so here.

Newton's Law and Its Applications to illiSol

The original illiSol was built around Newton's Law. Redesigned from a N-body computational program written in VPython, the first draft of illiSol computed the forces between the stationary and massive Sun and each of the planets using Newton's Universal Law of Gravitation. Unfortunately, this approach also highlights the fragility of planetary systems. Trying to initialize the planets into correct orbits was nearly impossible, and with actual values for mass, distance between bodies, orbital velocities, etc., the planets would

[6] Also paraphrased from Toeplitz

either be immediately sucked into the sun or escape the sun's pull and fly out of the system unhindered. Needless to say, this first model was abandoned after a couple weeks of studying it to no avail. The second draft of illiSol instead relied on a set of mathematical equations provided by Jet Propulsion Laboratory[6]. The third and current version still relies heavily on these equations as well. We will cover these equations in the next section.

Modern Methods of Computing Orbits

With mankind putting satellites, shuttles, stations, and probes into space, we need a way to keep track of not just planets in motion but the orbits of our man-made vehicles as well. This is possible with a minimum of six numbers, often referred to as the Orbital or Keplerian Elements. Three of these elements are angles that describe orientation in space with respect to an inertial reference frame. The next two describe the size and shape of the orbit - these are the semimajor axis, a, and the eccentricity, ϵ, of the ellipse. The last element is the epoch, which specifies the date at which we want to find our element. This is necessary because the orbital elements of any body changes over time due to slight perturbations, which must be accounted for by computing errors in each element given the date. Using these six elements for any body following Newton's Law, we can find the position of the body at any time.

There are slight variations on the Keplerian Elements used - this is because there are many ways to parameterize an ellipse in space. Of course, some description of the Keplerian elements is in order.

Standard Keplerian Elements[7]

While not necessarily a "standard" persay, this set of Keplerian Elements is one of the more common sets. The elements are

- I - Angle of Inclination [degrees]
- Ω - Longitude of Ascending Node [degrees]
- ω - Argument of Periapsis [degrees]
- ϵ - Eccentricity []
- a - Semi-major Axis [astronomical units]
- M - Mean Anomaly at the epoch [degrees]

In dealing with orbital elements, there are some important differences to understand, the first of which is the concept of the Celestial Sphere and the ecliptic plane. The Celestial Sphere[8] is the sphere on which the sun appears to move when viewed from Earth, as if Earth were at the center of the Celestial Sphere. The ecliptic plane is the plane that intersects the Celestial Sphere at the sun's path.

[7]This section as a whole is referenced from Montenbruck's and Pfleger's book[4]

[8]Note that this is not the same Celestial Sphere that Ptomely hypothesized literally rotated around the Earth.

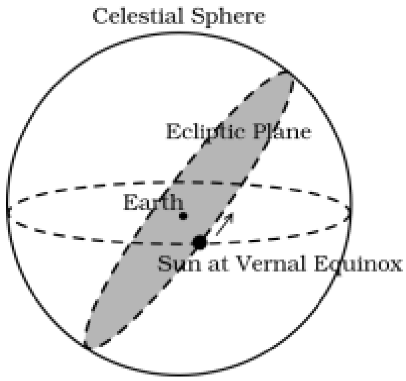

Figure E - The Celestial Sphere and the Ecliptic Plane

As shown in Figure E, the Vernal Equinox (at least in the Northern Hemisphere) occurs when the sun crosses what is known as the Celestial Equator, which coincides with the Earth's equator. At this point, the sun is shining directly on the Earth's equator, and an equinox occurs. This vernal equinox is important because it defines the reference direction for our solar system (this is also the direction of the First Point of Aries). This is illustrated in Figure F below in the context of the solar system - in terms of Cartesian coordinates, this direction is arbitrarily defined as the positive x axis, where the sun resides at the origin.

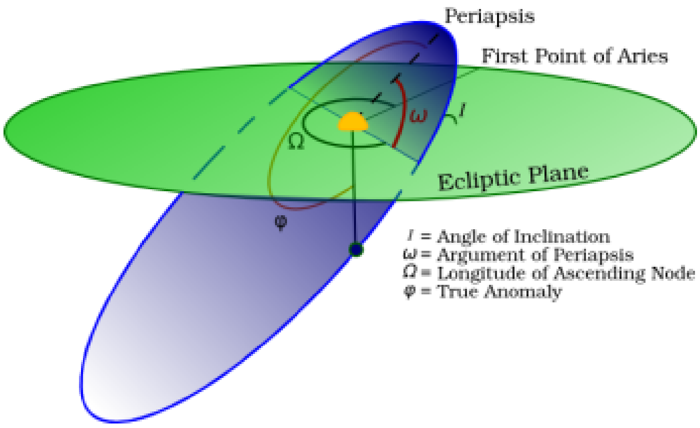

Figure F - Some of the Standard Orbital Elements

The longitude of ascending node (Ω) is the angle, measured counterclockwise from the reference direction (First Point of Aries), at which the the orbiting body crosses the ecliptic plane on its way "up". However, the definition of "up" in the context of the solar systen is difficult to define - in our case, all of the planets orbit counterclockwise in planes relatively close to being coplanar with the ecliptic. Therefore, we will define "up" with the right-hand rule - curl your fingers around the sun in the ecliptic in the direction that the planets orbit, and your thumb will point "up".

The angle of inclination (I) is exactly what it sounds like - the angle at which the orbital plane is inclined with respect to the ecliptic, measured at the longitude of the ascending node.

The argument of perihelion (ω) is the angle measured from the longitude of ascending node to the perihelion (or in the Earth's case, perigee). The general term for this point is the periapsis - it is the point of closest distance between the orbiting body and the central body in the orbit. Kepler's second law informs us that for elliptical orbits, the central body will lie at the focus of the ellipse. Therefore, referencing Figure D, the distance at the periapsis between the central body and the orbiting body obeying Newton's Law is simply $a - a\epsilon$, where for ellipses $0 < \epsilon < 1$.

The fourth variable illustrated in Figure F is the true anomaly (φ). This variable, while not a Keplerian Element, is familiar as it was used in the derivation of Newton's Law. This angle, measured counterclockwise from the periapsis to the orbiting body's current position, describes the body's position within the orbital plane. This angle is not calculated - instead, we approximate the eccentric anomaly (E in Figure G) by using the mean anomaly (M in Figure G).

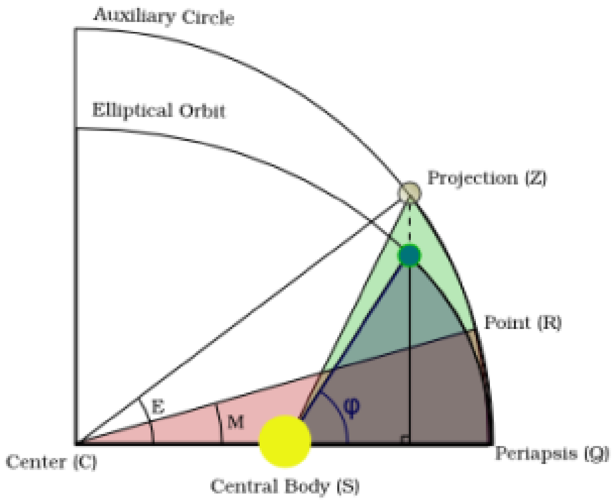

Figure G - Approximating the Eccentric Anomaly

In Figure G, the eccentricity of the elliptical orbit is greatly exaggerated. In reality, orbits in our solar system are near-circular, most with eccentricities less than .1. The notable exception is Mercury, with an eccentricity a little greater than .2.

The mean anomaly, shown as angle M in Figure G, represents the angle that the orbiting body would have traveled on the auxiliary circle. The mean anomaly is measured at "epoch," meaning that this is the variable element that incorporates time. The mean anomaly is chosen such that the area of its sector of the auxiliary circle (the red area) is equal to the area formed by $\angle QSZ$, which is shown in green. Note also that the green and blue areas are related as $|QSZ| = \frac{a}{b}|GSQ|$, if G is the point at which the orbiting body is currently positioned. We know, from Kepler's second law, that the blue area grows linearly with time, as the radius r sweeps out equal areas in equal times. Therefore, we see that the red area, the sector swept out by the mean anomaly, must also grow linearly in time, which, because M describes a central angle of a circle, mean that the mean anomaly itself grows linearly with time. Because of this, we are able to relate the mean anomaly M to the true anomaly φ, through use of an intermediate variable, the eccentric anomaly.

The eccentric anomaly, shown as angle E, describes the location of the orbiting body's projection onto the auxiliary circle, which is concentric with the ellipse and has a radius of

a, the magnitude of the semimajor axis of the ellipse. The projection of the orbiting body onto the auxiliary circle is obtained by extending the perpendicular from the semimajor axis to the current position of the orbiting body to the auxiliary circle, as is represented by the dashed line in Figure G. The eccentric anomaly is also not one of the orbital elements. However, it can be approximated using the mean anomaly with Kepler's Equation:

$$M = E - e \sin E$$

Unfortunately, the E cannot be isolated in this equation, so it is usually approximated by using numerical methods. There are many ways of doing this: there are various series solutions that allow for very accurate approximation. One may also note that

$$\lim_{\epsilon \to 0} E - \epsilon \sin E = E$$

so that for a circular orbit, the mean anomaly is equal to the eccentric anomaly, which is also equal to the true anomaly. Unfortunately for us, elliptical orbits require more effort to solve.

However, once we have obtained the eccentric anomaly, finding either the true anomaly or the position (in Cartesian coordinates) in the orbital plane is just one simple step away. The true anomaly is related to the eccentric anomaly as follows:

$$\varphi = \arccos\left(\frac{\cos E - e}{1 - e \cos E}\right)$$

Or, if you like, the radial distance from the orbiting body to the central body can be related to the eccentric anomaly as well:

$$r = a(1 - e \cos E)$$

The Jet Propulsion Lab (JPL) Documents[9]

illiSol uses a slightly varied set of Keplerian Elements to compute the positions of the eight major planets (data for Pluto, while available, was excluded). This set, referenced in the *Overview*, works with a set of equations provided by the Jet Propulsion Lab at Cal Tech, available on their website to any and all. The guide, written by E.M. Standish, delivers a thorough yet easy-to-use walkthrough for approximating planetary positions (insofar as exceptional accuracy is not required), as well as tables of Keplerian Elements and their respective error offsets, which are observationally derived, for the planets of interest. The illiSol "engine" implements the set of equations in C, giving viewers a relatively accurate visualization of the Solar System.

The JPL Documents use the following Keplerian Elements (the * indicates a difference from the standard elements from the last section):

- I - Angle of Inclination [degrees]
- Ω - Longitude of Ascending Node [degrees]
- ϖ* - Longitude of Periapsis [degrees]
- ϵ - Eccentricity []

[9] All equations and data presented in this section are taken from E.M. Standish's paper [6]

- a - Semi-major Axis [astronomical units]
- $L*$ - Mean Longitude [degrees]

The longitude of periapsis is very similar to the aforementioned argument of periapsis - in fact, the longitude of periapsis is simply the sum of the longitude of ascending node and the argument of periapsis:

$$\varpi = \Omega + \omega$$

This angle can be achieved by starting at the reference direction (usually the Vernal Equinox) and moving counter-clockwise around the central body on the ecliptic plane until the ascending node is reached, at which point the angle continues up onto the orbital plane until the periapsis is reached.

The mean longitude, similarly, is simply a re-incarnation of the mean anomaly:

$$L = M + \varpi + X$$

where X is a set of terms that must be added when dealing with relatively distant dates (i.e. before 1800 AD and after 2050 AD).[10] Furthermore, the extra terms will only compensate accurately for dates between 3000 BC and 3000 AD.[11]

The mean longitude represents the position of the orbiting body under the conditions that its orbit is circular and coplanar with the ecliptic. Because ϖ is constant over relatively small time periods, the mean longitude always leads the mean anomaly by ϖ degrees. Therefore, we can use the mean longitude to compute the mean anomaly, and therefore, the eccentric anomaly as well using Kepler's Equation.

The following is an excerpt from the JPL Documents showing the Keplerian Elements and their respective errors over time for the inner planets between 1800 AD and 2050 AD:

	a [au, au/cty]	e [, /cty]	I [deg, deg/cty]	L [deg, deg/cty]	ϖ [deg, deg/cty]	Ω [deg, deg/cty]
Mercury	0.38709843	0.20563661	7.00559432	252.25166724	77.45771895	48.33961819
	0.00000000	0.00002123	-0.00590158	149472.67486623	0.15940013	-0.12214182
Venus	0.72332102	0.00676399	3.39777545	181.97970850	131.76755713	76.67261496
	-0.00000026	-0.00005107	0.00043494	58517.81560260	0.05679648	-0.27274174
EM Bary	1.00000018	0.01673163	-0.00054346	100.46691572	102.93005885	-5.11260389
	-0.00000003	-0.00003661	-0.01337178	35999.37306329	0.31795260	-0.24123856
Mars	1.52371243	0.09336511	1.85181869	-4.56813164	-23.91744784	49.71320984
	0.00000097	0.00009149	-0.00724757	19140.29934243	0.45223625	-0.26852431

Note how as the planets get farther away from the sun, the error values of their mean longitudes (L) decrease. This is because in its very simplest form, the mean longitude is a way of determining how many degrees the radius vector from the sun to the planet in question has rotated in the last time step. To illustrate this, we will look at a day in the orbit of Earth.

[10] While the ability to switch to the distant-date equations is lurking in illiSol, it is never actually used.

[11] illiSol can run until shortly after 600000 AD, at which point it usually crashes. This, however, says nothing about the accuracy from 2050 AD to 600000 AD.

Earth has an orbital period of about 365.25 days, which means that in one century, about 36,525 Earth-days have passed. Let us consider the very beginning of January, 2000 AD, roughly one day past J2000 (the reference point by which dates are calculated):

- We are $\frac{1}{36525}$ centuries past J2000.
- The offset of the mean longitude of Earth per century is 35999.37306329°
- The offset of the longitude of perihelion of Earth per century is 0.31795260°
- The amount of offset in the mean longitude in one day of the Earth's orbit is

$$\frac{1}{36525} \cdot 35999.37306329^\circ = 0.985609119^\circ$$

- However, the amount of offset in the longitude of perihelion in one day of the Earth's orbit is only

$$\frac{1}{36525} \cdot 0.31795260^\circ = 8.70506776 \times 10^{-6^\circ}$$

  which for our purposes, is negligible.

- Therefore, since the mean anomaly is equal to the difference of the mean longitude and the longitude of perihelion, we find the *difference* in mean anomaly from one day to the next is equal to the offset in the mean longitude for that day:

$$\Delta M = M_1 - M_0 = (L_{adjusted} - \varpi_{adjusted}) - (L_{base} - \varpi_{base})$$

  but since

$$\varpi_{adjusted} \approx \varpi_{base}$$
$$\Delta M = L_{adjusted} - L_{base} = L_{offset}$$

  we see that

$$\Delta M = 0.985609119^\circ$$

This makes perfectly good sense - the Earth's orbit has a low eccentricity, and is almost circular, meaning that the mean anomaly is about equal to the true anomaly φ. Since the Earth's orbital period is 365 days and there are 360° in a circle, we would expect the change in the true anomaly to be slightly less than 1° per day. This also explains why the offsets in mean longitude decrease as we move further away from the sun - the outer planets have much longer orbital periods, and thus the rate of change of their mean anomaly should be much smaller as well.

Of course, that which is stated as negligible becomes nonnegligible over long periods of time - it is in this way that orbits slowly change. However, the source of this change lies in perturbation theory, and is far beyond the scope of this paper.

Now that we have data by which to derive the mean anomaly of the planet in question, the next step is to find the eccentric anomaly using this information. As stated before, Kepler's Equation does not have a known closed-form solution. However, the JPL Documents provide an approximate solution that requires Newton's Method.[12] Though not as accurate as some series solutions, Newton's Method provides a quick way to approximate the eccentric anomaly in a way that is easily implemented computationally. The actual method for solving Kepler's Equation in this way is not included here.[13]

There are two ways to proceed from here - one could calculate the true anomaly of the orbit at the requested epoch or one could directly calculate the orbiting body's position in the orbital plane. Both methods require the eccentric anomaly, and for our purposes, calculating the Cartesian coordinates for the orbiting body is far more useful.

We can find the position of the orbiting body in its own orbital plane using the following equations:

$$x_{orbit} = a(\cos E - \epsilon) \quad y_{orbit} = a\sqrt{1-\epsilon^2}\sin E \quad z_{orbit} = 0$$

where the x axis is collinear with the line from the periapsis to the central body. We can also see that these can be reconciled with the equation relating the eccentric anomaly to the radius from the central body to the orbiting body.

$$x_{orbit}^2 + y_{orbit}^2 = a^2\cos^2 E - 2a^2\epsilon\cos E + a^2\epsilon^2 + a^2(1-\epsilon^2)\sin^2 E$$
$$x_{orbit}^2 + y_{orbit}^2 = a^2(\cos^2 E + \sin^2 E) - 2a^2\epsilon\cos E + a^2\epsilon^2(1-\sin^2 E)$$
$$x_{orbit}^2 + y_{orbit}^2 = a^2(1 - 2\epsilon\cos E + \epsilon^2\cos^2 E)$$

Similarly, if we square our equation for r in terms of eccentric anomaly, we get

$$r^2 = (a - a\epsilon\cos E)^2$$
$$r^2 = a^2 - 2a^2\epsilon\cos E + a^2\epsilon^2\cos^2 E$$
$$r^2 = a^2(1 - 2\epsilon\cos E + \epsilon^2\cos^2 E)$$

Therefore, we have confirmed that

$$x_{orbit}^2 + y_{orbit}^2 = r^2$$

as it should be.

Generally, we want to obtain the heliocentric coordinates in the ecliptic planes. The manner in which to proceed is to rotate our axes to account for the angle of inclination, the longitude of ascending node, and the argument of periapsis. The JPL Documents provide predetermined equations to do this for us:

$$x_{ecliptic} = (\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos I)x_{orbit} - (\sin\omega\cos\Omega + \cos\omega\sin\Omega\cos I)y_{orbit}$$
$$y_{ecliptic} = (\cos\omega\sin\Omega - \sin\omega\cos\Omega\cos I)x_{orbit} - (\sin\omega\sin\Omega - \cos\omega\cos\Omega\cos I)y_{orbit}$$
$$z_{ecliptic} = (\sin\omega\sin I)x_{orbit} + (\cos\omega\sin I)y_{orbit}$$

However, because the Syzygy graphics environment relies on OpenGL[14] there is an easier way to do this.

[12] This is fitting, given that this ultimately boils down to Newton's Law of Gravitation as well.

[13] This can be found in the JPL Documents.

[14] The Syzygy framework sets up the application to run easily on a cluster of computers, each with its own copy of the application. Syzygy synchronizes each of the computers, but the display function is usually pure OpenGL. This usually allows for easy conversion from OpenGL applications to Syzygy applications (most of the changes occur in the initializing functions) while retaining all of OpenGL's capabilities.

The OpenGL Modelview Matrix

The OpenGL Matrix is how OpenGL keeps track of the properties of its scene - it does not keep track of the objects in the scene. There are three basic matrix stacks in OpenGL - `GL_MODELVIEW`, `GL_PERSPECTIVE`, and `GL_TEXTURE`. `GL_MODELVIEW` is the default, in which all matrices on the stack are applied to the model in the three-dimensional scene. `GL_MODELVIEW` is the matrix mode we want, so we don't have to explicitly set it[1]. The code on the next page illustrates how we can use OpenGL's matrix functionality to our advantage.

Note the call to OpenGL in line 2. This call pushes a copy of the current descriptive modelview matrix onto the top of the stack. We can consider this our "scratch paper" - any matrix multiplications applied to the matrix on the top of the stack will be lost when `glPopMatrix()` is called. Conversely, any matrix multiplications applied to matrices underneath the top of the stack are encoded into the matrix on the top of the stack.

```
1   /* Assume all variables are defined */
2   glPushMatrix();
3           glTranslatef(x0, y0, z0);
4           /* Draw Sol Here */
5           ...
6           /* End of drawing Sol */
7           /* Now go to where we want to draw the planet,
8            * relative to Sol */
9           glPushMatrix();
10                  /* Rotate our axes */
11                  glRotatef(ascNode, 0, 0, 1);
12                  glRotatef(inc, 1, 0, 0);
13                  glRotatef(argPeri, 0, 0, 1);
14                  /* Translate to the planet's orbital plane coordinates */
15                  glTranslatef(x1, y1, 0);
16                  /* Draw the planet here */
17                  ...
18                  /* End of drawing the planet*/
19          glPopMatrix();
20  glPopMatrix();
```

Figure H - OpenGL code illustrating articulation[15]

Applications in the Orrery

We can use this method of pushing matrices onto the modelview stack to nest our orbits as deeply as we like. The basic program structure might look something like this:

- Compute the position of each orbiting body in its own orbital plane.
- Compute the position of each body's satellites in its own orbital plane around the orbiting body.[16]

[15] Articulation is the method of using the OpenGL matrix stack to place things correctly in the world. See "Metarealistic Renderings of Real-time Computer Animations" by Dr. George Francis.

[16] This greatly simplifies calculations - determining the heliocentric coordinates of geocentric orbiting body can be quite difficult.

- Draw the central body at the origin.
- Push a copy of the modelview matrix onto the stack.
  - Rotate the axes properly to account for the angle of ascending node, angle of inclination, and argument of periapsis.
  - Translate to the previously-computed orbiting body's position.
  - Draw the orbiting body at what is now the origin of the current coordinate system.
  - For each satellite of the orbiting body, push a copy of the modelview matrix onto the stack, and repeat the process of rotation, translation, and drawing, using the previously computed properties for each satellite of the orbiting body. After completing this process for each satellite, pop the current modelview matrix from the top of the stack to return to the orbiting body.
  - Pop the current modelview matrix from the top of the stack to return to the central body.
- Repeat the previous step for each orbiting body.
- Repeat the entire process again for the next timestep.

As stated before, in the context of OpenGL there is another way to rotate the coordinate system than using equations. With three calls to `glRotatef` or `glRotated`, we can configure the modelview matrix to position the orbiting body using only the orbital plane coordinates.

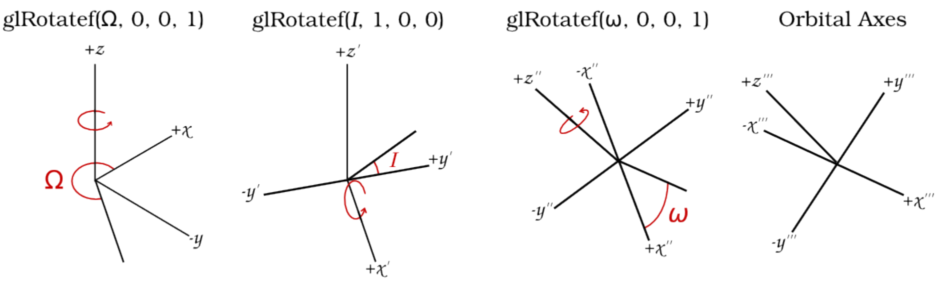

Figure I - Rotating Axes Using Keplerian Elements

Figure I shows how three successive matrix multiplications can rotate us from the ecliptic plane[17] (or any reference plane for that matter) to the orbital plane given the correct Keplerian Elements of the elliptical orbit. This is mathematically intuitive - any plane in $\mathbb{R}^3$ can be described by two angles, but a third angle must be used to describe the reference angle from which all angles in the orbital plane are measured. In the case of elliptical orbits, the longitude of ascending node and the angle of inclination describe the plane in $\mathbb{R}^3$, whereas the argument of periapsis is used to align the x axis of the new coordinate axes with the periapsis.[18] In effect, of course, this will accomplish precisely the same goal as using the JPL Documents as they are.

[17] Recall from earlier that angles in the ecliptic plane are measured from the Vernal Equinox, the direction of the First Point of Aries.

[18] Remember that the equations for computing the position of the orbiting body within its own plane assume that the $+x$ axis is aligned from the central body toward the periapsis.

Actual placement of the planet is quite simple once the coordinate axes are properly aligned in the orbital plane. The call to `glTranslatef` in line 15 of the code on page 19 is yet another matrix multiplication to position the modelview at the point `(x1, y1, 0)` where `x1` and `y1` are the x and y coordinates in the orbital plane, respectively, that we computed earlier. From this point, we can draw the planet and, if we have enough data, we can apply the same process to position the satellites of the planet.

Summary, Conclusion & Acknowledgements

Summary

Beginning with the work of Kepler, we studied the derivation of Newton's Universal Law of Gravitation and its compatibility with Kepler's observationally derived laws of planetary motion. Duly noted, however, was the fact that Newton's Law is inadequate for full simulation of the Solar System. Instead, I described how I turned to current mathematical methods for computing the orbits of the major planets to govern the orrery.

The aforementioned methods use what are known as the Keplerian Elements, and using these and a set of equations provided by the Jet Propulsion Lab, we can obtain not only coordinates of the planet in its orbital plane at a certain time, but we can also find its heliocentric coordinates. Furthermore, I showed that in the context of OpenGL programming, this last step is unnecessary, and can instead be implemented with native OpenGL functions.

Conclusion

But remember that the ultimate goal of illiSol is to provide a better visualization for celestial mechanics, working around the issues of both time and scale. It accomplishes this using the relatively accurate equations mentioned above and orbital element data that has been observationally fit by JPL to describe the orbits of the major planets.

The purpose of the paper was to extend the visualization to encompass the mathematical, attempting to state in simplest terms the real science and mathematics behind the equations used in illiSol, as well as to provide a brief but applicable introduction to celestial mechanics. As a newborn to the field, I would have appreciated such a synoptical guide. Likewise, this guide is meant to be critiqued and improved upon to help to increase its efficacy.

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