HPP, FHP, and FHCP Implementation

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Computational Fluid Dynamics with CA

Visualization

In virtually all the papers read, the lattice grid is viewed as a vector field. This is done for several reasons:

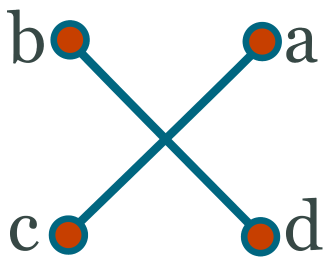
- It's easy to visualize on paper whether the algorithm is working.
- It's the simplest to implement.

In NKS, for example, Wolfram uses a 96×96 subgrid to approximate a vector direction on a 6 million lattice grid. Previously in his 1987 paper, Wolfram used .... for a 4600(?) lattice grid.

HPP Model

We represent each lattice with 4 vectors, each containing is stored in its own matrix. This means that `a[i][j]` contains the `a` vector component of the lattice on the i^{th} row and j^{th} column. With this representation in place, we perform the following to compute the next frame.

- Resolve collisions — the collisions are performed and stored in a temporary collision matrix.
- Propagation — the propagation is performed and values are copied from the temporary collision matrix into the `a`, `b`, `c`, and `d` matrices.



Collision

Four temporary matrices are created and are called `K_a`, `K_b`, `K_c`, and `K_d` to signify which direction they correspond to. The vector `K_a[i][j]` is the updated based on the HPP collision rules.

One can represent the collision rules as a series of binary statements. The best example is found in[1] where the authors represent the collision rules as a binary statement. Examples of the same binary statement can be found in other papers, however (**need to record them**)

\$ $a = a \otimes [(a \wedge c \wedge \neg(b \vee d)) \vee (b \wedge d \wedge \neg(a \vee c))]$ \$

Where the logic operator symbols are

- $\neg$ is the *not* operator
- $\otimes$ is the *xor* operator
- $\vee$ is the *or* operator
- $\wedge$ is the *and* operator

In `C`, the following code implements the above rule:

```
for (x = 0; x < LATTICE_WIDTH; x++) {
    for (y = 0; y < LATTICE_HEIGHT; y++) {
        change = (a[x][y] & c[x][y] & ~(b[x][y] | d[x][y])) |
                 (b[x][y] & d[x][y] & ~(a[x][y] | c[x][y]));

        K_a[x][y] = a[x][y] ^ change;
        K_b[x][y] = b[x][y] ^ change;
        K_c[x][y] = c[x][y] ^ change;
        K_d[x][y] = d[x][y] ^ change;
    }
}
```

[1] Wolf-Gladrow, Dieter. Lattice-Gas Cellular Automata and Lattice Boltzmann Models: an Introduction. Santa Clara: Springer-Verlag TELOS, 2000. (pdf)

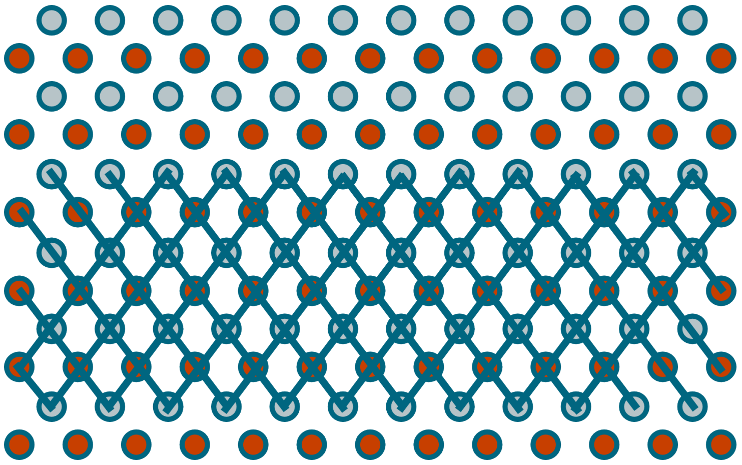
The boolean expression allows us to perform the computation needed without the use of `if-else` statements. It should be also noted that we will never go outside the bounds of the array, the same cannot be said about the propagation step. Also note that the expression

```
(a1[x][y] & c1[x][y] & ~(b1[x][y] | d1[x][y])) |
(b1[x][y] & d1[x][y] & ~(a1[x][y] | c1[x][y]))
```

Is repeated four times in the binary representation, which is why we store it in a separate value to reduce three computations.

Propagation

The HPP propagation step is the process of moving velocity vectors in the same direction they are moving in after the collisions are resolved. Because of the way we oriented our grid, we made this step unnecessarily complicated. Also it's not clear if our representation has any advantages over a regular square grid. Future implementation (if one is needed) would use a regular square grid.



We use the following code to propagation the vectors in the lattice

```
a[i][j] = K_a[i-1][j-1];
b[i][j] = K_b[i-1][j+1];
c[i][j] = K_c[i+1][j+1];
d[i][j] = K_d[i+1][j-1];
```

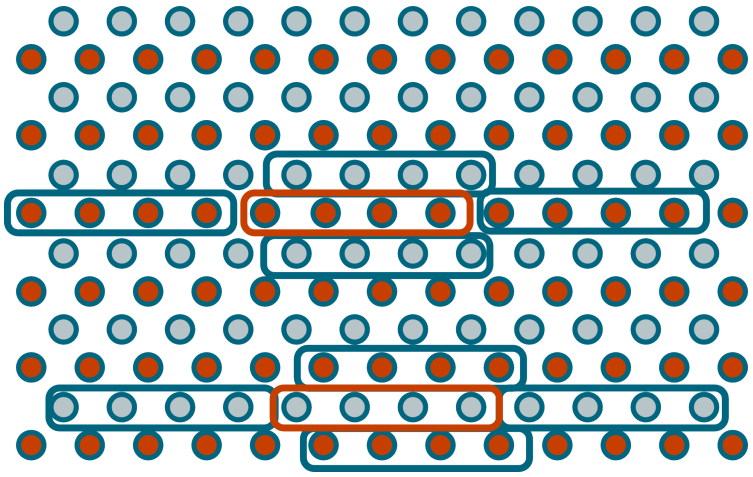
it should be evident that we will run out of bounds if we run the above piece of code over the width of the lattice array. Since we do not want to enforce any boundary conditions at this stage of the process, we place our lattice on a toroidal surface. This can be easily implement in `C` by noting that the following way to access and store array elements creates the toroidal surface:

```
t[(i+n + LATTICE_HEIGHT) % LATTICE_HEIGHT][(j+m + LATTICE_WIDTH) % LATTICE_WIDTH]
```

and will also always be in the bounds of the matrix representation.

Optimizations

Since C has no differentiation between a boolean value and a short, we can overload a short variable to contain multiple vector values. The boolean expression does not change, but as you see from the figure below, the propagation step has to.



The Code

There are a few differences between our implementation and the one described above:

1. The variables are called `c1`, `c2`, `c3`, and `c4` rather than `a`, `b`, `c`, and `d`. The temporary collision arrays have also been renamed to `k1`, `k2`, `k3`, and `k4`.
2. Our computation of blocks of vectors at a time forces us to use checkerboard propagation.

The crux of the code for HPP is provided bellow

```
int x, y;
short change;

// Resolve Collisions
```

```
for (x = 0; x < LATTICE_WIDTH; x++) {
    for (y = 0; y < LATTICE_HEIGHT; y++) {
        change = (a1[x][y] & c1[x][y] & ~(b1[x][y] | d1[x][y])) |
                 (b1[x][y] & d1[x][y] & ~(a1[x][y] | c1[x][y]));

        a2[x][y] = a1[x][y] ^ change;
        b2[x][y] = b1[x][y] ^ change;
        c2[x][y] = c1[x][y] ^ change;
        d2[x][y] = d1[x][y] ^ change;
    }
}
// Propagate
for (x = 1; x < LATTICE_WIDTH - 1; x++) {
    for (y = 1; y < LATTICE_HEIGHT - 1; y += 2) {
        a1[x][y] = (a2[x][y - 1] >> 1) + (a2[x - 1][y - 1] << LAST);
        b1[x][y] = b2[x][y - 1];
        c1[x][y] = c2[x][y + 1];
        d1[x][y] = (d2[x][y + 1] >> 1) + (d2[x - 1][y + 1] << LAST);

        a1[x][y + 1] = a2[x][y];
        b1[x][y + 1] = (b2[x][y] << 1) + (b2[x + 1][y] >> LAST);
        c1[x][y + 1] = (c2[x][y + 2] << 1) + (c2[x + 1][y + 2] >> LAST);
        d1[x][y + 1] = d2[x - 1][y + 2];
    }
}
```

CUDA

Never implemented.

FHP Model

Collision

Propagation

The Code

Optimizations

CUDA

FCHC Model

In the *Face Centered Hyper Cubic* model, you have 24 vectors. 16 correspond to the vertexes of the hyper cube while the other 8 correspond to the center of the squares in the hyper cube.

Reduction

Since you cannot store the entire table, you consider only states that have a normalized momentum. The coordinates of the momentum q are defined by

$\$q_\alpha = \sum_{i=0}^{24} S_i C_{i\alpha}\$$

where S_i is a boolean variable signifying the presence or absence of a particle with velocity C_i. The momentum is normalized if the coordinates satisfy

$\$q_1 > q_2 > q_3 > q_4 > |q_1 - q_2 - q_3|\$$

Different isometries are then applied if one of the following conditions are satisfied in the following sequence:

- if $q_1 < 0$, then S_1 is applied to q
- if $q_2 < 0$, then S_2 is applied to q
- if $q_3 < 0$, then S_3 is applied to q
- if $q_4 < 0$, then S_4 is applied to q
- if $q_1 < q_2$, then P_{12} is applied to q
- if $q_3 < q_4$, then P_{34} is applied to q
- if $q_1 < q_3$, then P_{13} is applied to q

- if $q_2 < q_4$, then P_{24} is applied to q
- if $q_2 < q_3$, then P_{23} is applied to q
- if $q_1 + q_4 < q_2 + q_4$, then Σ_1 is applied to q
- if $q_1 < q_2 + q_3 + q_4$, then Σ_2 is applied to q

Where

- S_α is the symmetry with respect to the plane $x_\alpha = 0$
- $P_{\alpha\beta}$ is the symmetry with respect to the plane $x_\alpha = x_\beta$
- Σ_1 is the symmetry with respect to the plane $x_1 + x_4 = x_2 + x_3$
- Σ_2 is the symmetry with respect to the plane $x_1 = x_2 + x_3 + x_4$