MATHEMATICAL FINANCE

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Let's start out with a simple situation. Suppose that I am willing to write a financial contract and sell it to you. The financial contract is as follows. In three months, this contract allows you (but does *not* force you) to buy a share of Microsoft (MSFT) from me for \$70 one month from now (August 11). The current price of MSFT is \$65. How much should you pay for this contract, or alternately how much should I charge for the contract?

Remark 0.1. The above is a *European call.* It is a call in that it allows the holder to purchase a share. It is European in that it can be *exercised* only at *expiry* (August 11). We refer to \$70 in the above example as the *strike price.*

Clearly we need some more components to our model. Let's assume a *binomial model*; that on August 11, the stock can be only at two prices. For specificity, let's assume that on August 11, must be either at \$80 or at \$55. Let's also assume that the monthly interest rate is 2%. Note that in our model, if we define

$$S \overset{\text{def}}{=} \$65, \qquad u \overset{\text{def}}{=} \frac{\$80}{\$65} = 1.23 \qquad d \overset{\text{def}}{=} \frac{\$55}{\$65} = .85, \qquad \text{and} \qquad R \overset{\text{def}}{=} 1.02$$

then we have that the current stock price is S, and it either goes up to uS or down to dS. If we put a dollar in the bank, then in one month we get $\$R$ out in one month. This will turn out to be a better formulation of things.

Let's next understand the value of the option on August 11. Assume that you have bought the option. If the stock is at \$80, then you would of course exercise the option and resell your stock on the market. You would get $V_u \overset{\text{def}}{=} \$80 - \$70 = \$10$. If the stock is at \$55, then the option is worthless, so let's define $V_d \overset{\text{def}}{=} 0$. A moment's thought reveals that *if the price of the stock at expiry is p, then the price of a call with strike K is* $\max\{p-K, 0\}$.

Let's next set up the following diagram. We know V_u and V_d. We want to find the price of the option today; i.e., we want to find V. Let's consider a portfolio.

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Given at the IlliMath2001 seminar on July 11th, 2001. This is not original research, but rather an elementary discussion of the binomial method in mathematical finance.

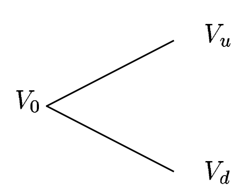

FIGURE 1. One-Period Model for Option

Let's buy the option and Δ units of stock. The value of this portfolio now us $V_0 + \Delta S$. If the stock goes up, the value of our portfolio on August 11 is

$$V_u + \Delta u S = \$10 + \Delta \$80$$

and if the stock goes down, the value of our portfolio on August 11 is

$$V_d + \Delta d S = \$0 + \Delta \$55.$$

Let's next make our key observation. *We can arrange things to hedge away risk in our portfolio.* Namely, let's arrange things so that

$$V_u + \Delta u S = V_d + \Delta d S;$$

i.e., let's choose

$$\Delta = -\frac{V_u - V_d}{S(u-d)} = -\frac{\$10 - \$0}{\$80 - \$55} = -0.4.$$

More precisely, buying -0.4 share of stock corresponds to *shorting* 0.4 of a share of stock. Since the value of our portfolio on August 11 is completely certain, so is its value now (via the interest rate). We must have

$$V_0 + \Delta S = \frac{V_u + \Delta u S}{R} = \frac{V_d + \Delta d S}{R}.$$

Putting in specific values, we get that

$$\begin{aligned} V_0 + \Delta S &= V_0 - 0.4(\$65) = V_0 - \$26 \\ \frac{V_u + \Delta u S}{R} &= \frac{\$10 - 0.4\$80}{1.02} = -\$21.57 \\ \frac{V_d + \Delta d S}{R} &= \frac{\$0 - 0.4\$55}{1.02} = -\$21.57 \end{aligned}$$

Thus, we must have that

$$V_0 = \$26 - \$21.57 = \$4.43.$$

Symbolically, we have

$$\begin{aligned} V_0 &= \frac{V_u + \Delta u S}{R} - \Delta S = \frac{V_u}{R} + \Delta S\left(\frac{u}{R} - 1\right) \\ &= \frac{V_u}{R} - \frac{V_u - V_d}{u-d}\left(\frac{u}{R} - 1\right) = \frac{V_u}{R} - \frac{V_u - V_d}{R}\frac{u-R}{u-d} \\ &= \frac{V_u}{R}\left\{1 - \frac{u-R}{u-d}\right\} + \frac{V_d}{R}\frac{u-R}{u-d} = \frac{V_u \varrho_u + V_d \varrho_d}{R} \end{aligned}$$

where we have defined the *risk-neutral probabilities*

$$\varrho_u \stackrel{\text{def}}{=} \frac{R-d}{u-d} \qquad \text{and} \qquad \varrho_d \stackrel{\text{def}}{=} \frac{u-R}{u-d}.$$

In our case, we have

$$\varrho_u \stackrel{\text{def}}{=} \frac{1.02 - .85}{1.23 - .85} = 0.447 \qquad \text{and} \qquad \varrho_d = \frac{1.23 - 1.02}{1.23 - .85} = 0.553.$$

Note that

$$\frac{(0.447)\$10 + (0.553)\$0}{1.02} = \$4.38$$

(the difference between 4.38 and 4.43 is a reasonable 1% numerical error).

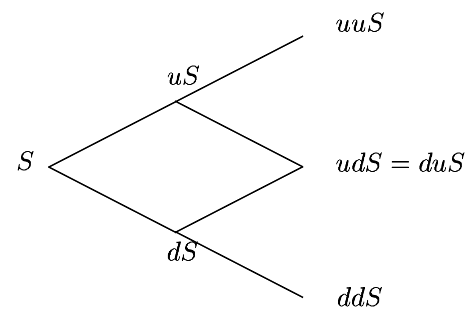

FIGURE 2. Two-Period Stock Model

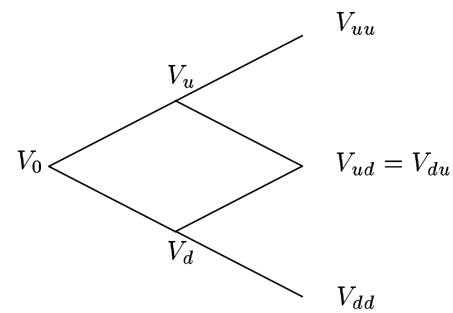

FIGURE 3. Two-Period Option Model

Remark 0.2. Let's organize our thoughts as follows. We were able to compute the price of the option at the end of the period since the end of the period was expiry. Once we knew the price of the option at the end of the period, we could compute it at the beginning.

Let's consider now a more complicated model. Assume that in one month, the stock can still either go up by 23% or down by 10%, and the monthly interest rate is 2%. Assume, however, that the option expires on September 11, rather than August 11.

Then we have the following tree models for the stock price and the option prices. Let's next set up the following tree for the stock prices. Here we have

$$S = \$65 \quad uS = \$80 \quad dS = \$55 \quad uuS = 98.34$$
$$udS = duS = 67.95 \quad \text{and} \quad ddS = 46.96.$$

Let's also set up a model for the option prices. At expiry, the option prices will be

$$V_{uu} = \max\{\$98.34 - \$70, \$0\} = \$28.34$$
$$V_{ud} = V_{du} = \max\{\$67.95 - \$70, \$0\} = \$0$$
$$V_{dd} = \max\{\$46.96 - \$70, \$0\} = \$0.$$

The option prices at the internal nodes are calculated as above; they are

$$V_u = \frac{(0.447)\$28.34 + (0.553)\$0}{1.02} = \$12.42$$

$$V_u = \frac{(0.447)\$0 + (0.553)\$0}{1.02} = \$0.$$

Finally, the option price now is

$$V_0 = \frac{(0.447)\$12.42 + (0.553)\$0}{1.02} = \$5.44.$$

To justify this last calculation, recall Remark 0.2. In general, we work from expiry backwards.

There are many more complications which one could add. An *American* option is one which can be exercised at any time before expiry. More complicated payoff functions (the value of the option at expiry) could be considered. One could consider path-dependent options (in which $V_{ud} \neq V_{du}$; examples are lookbacks and Russian options). One could also price bonds and currency derivatives using the binomial model.

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